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ABSTRACT

This proceedings of the annual conference of the International Group for the Psychology of Mathematics Education (PME) includes the following research papers: "Logo et Symetrie Centrale" (E. Gallou-Dumiel); "About Continuous Operator Subconstruct in Rational Numbers" (J. Gimenez); "Constructivist Epistemology and Discovery Learning in Mathematics" (G.A. Goldin); "Comparative Analysis of Two Arithmetic Situations" (M.G. Grossi); "Understanding and Discussing Linear Functions in Situations: A Developmental Study" (J.L. Gurtner); "Low Mathematics Achievers' Test Anxiety" (R. Hadass); "Proofs That Prove and Proofs That Explain" (G. Hanna); "Fishbein's Theory: A Further Consideration" (G. Harel, M. Behr, T. Post, & R. Lesh); "They're Useful: Children's View of Concrete Materials" (K. Hart & A. Sinkinson); "Children's Individuality in Solving Fraction Problems" (K. Hasemann); "Aspects of Declarative Knowledge on Control Structures" (K. Haussmann & M. Reiss); "A Conceptual Analysis of the Notion of Length and Its Measure" (B. Heraud); "The Kindergartners' Understanding of Cardinal Number: An International Study" (N. Herscovics & J.C. Bergeron); "Learning about Isosceles Triangles" (J. Hillel); "Construction of Functions, Contradiction and Proof" (F. Hitt); "A Logo-Based Microworld for Ratio and Proportion" (C. Hoyles, R. Noss, & R. Sutherland); "The Facilitating Role of Table Forms in Solving Algebra Speed Problems: Real or Imaginary?" (R. Hoz & G. Harel); "The Learning of Plane Isometries from the Viewpoint of the Van Hiele Model" (A. Jaime & A. Gutierrez); "Representation and Contextualization" (C. Janvier); "To Inculcate versus to Elicit Knowledge" (B. Jaworski); "Van Hiele Levels and the Solo Taxonomy" (M. Jurdak); "A Perspective on Algebraic Thinking" (C. Kieran); "A Structural Conceptual Model for Investigating Some Cognitive Aspects of Problem-Solving" (N. Krumholtz); "The Effect of Setting and Numerical Content on the Difficulty of Ratio Tasks" (D. Kuchemann); "Satisfaction and Regret about the Choice of Math" (H. Kuyper & W. Otten); "Intrinsic versus Euclidean Geometry: Is the Distinction Important to Children Learning with the Turtle?" (C. Kynigos); "Mathophobia: A Classroom Intervention at the College Level" (R. Lacasse & L. Gattuso); "Le Micro-Ordinateur, outil de Revelation et d'Analyse de Procedures dans de Courtes Demonstrations de Geometrie" (A. Larher & R. Gras); "Gender Differences in Mathematics Learning Revisited" (G.C. Leder); "La Resolution de Problemes dans l'Enseignement des Mathematiques: Compte Rendu d'une Experience aupres d'Enseignants du Primaire" (G. Lemoyne & F.

Conne); "Strategies Used by 'Adders' in Proportional Reasoning" (F.L. Lin); "Canonical Representations of Fractions as Cognitive Obstacles in Elementary Teachers" (L. Linchevsky & S. Vinner); "Using Concept Maps to Explore Students' Understanding in Geometry" (H. Mansfield & J. Happs); "Mental Images: Some Problems Related to the Development of Solids" (M.A. Mariotti); "The Role of the Figure in Students' Concepts of Geometric Proof" (W.G. Martin & G. Harel); "The Inner Teacher, The Didactic Tension, and Shifts of Attention" (J.H. Mason & P.J. Davis); "Lecture et Construction de Diagrammes en Batons dans le Premier Cycle de l'Enseignement Secondaire Francais" (S. Maury, M. Janvier, & J. Baille); "Cooperative Group Learning of Geometric Proof Construction: A Classroom Assessment" (L. MacRae & B. Harrison); "Comparing Experts' and Novices' Affective Reactions to Mathematical Problem Solving: An Exploratory Study" (D. McLeod, W. Metzger, & C. Craviotto); and "The Development of Children's Concepts of Angle" (M.C. Mitchelmore). (MKR)

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Volume 2

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PME 13

Volume 2

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LOGO ET SYMETRIE CENTRALE

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ABSTRACT. — Point symmetry is a transformation taught in the secondary shool in France which involves a problem of orientation. A comparison between the construction of the image of an angle in the case of a point symmetry and in the case of a reflection seems to be useful. For this we undertook the construction of a learning of point symmetry in the same LOGO environment than the one we realized before for the learning of reflection. We explain the choice of the LOGO environment, of the tasks, of the figures and we give the results of the experimentation in three classes in Grenoble (France).

1-Introduction,problématique.- La symétrie centrale est une transformation qui change l'orientation d'un solide dans un espace de dimension 3 et qui, dans un plan transforme une figure en une figure superposable. La conservation des angles du plan semble un des points essentiels de cette notion.

Il nous a semblé judicieux de construire une séquence d'apprentissage qui par le choix des tâches, des consignes, le dispositif, soit analogue à celle réalisée pour la symétrie orthogonale pour permettre à l'élève d'établir des mises en relation de ces deux notions et pour l'amener à déterminer la spécificité de chacune d'entre elles. La détermination du symétrique d'un angle nous semble l'activité fondamentale pour la symétrie centrale comme pour la symétrie orthogonale.

Nous avons donc choisi un dispositif identique à celui de la séquence d'apprentissage réalisé pour la symétrie orthogonale: un micro - ordinateur avec la liste restreinte des commandes LOGO: AV n , RE n , TD n , TG n , ORIGINE , LC , BC , GOMME , FINGOMME et VE.

En effet ce dispositif réalise un "micro monde" où la procédure suivante appelée procédure de tracé par segments initialisés est favorisée. Cette procedure consiste à tracer une figure formée de segments juxtaposés en indiquant à chaque sommet l'angle dont doit tourner la tortue puis la longueur du côté à tracer. Ce "micro-monde" est différent de celui de la géométrie de la règle et du compas où sont favorisées les propriétés d'incidence et les reports de longueurs mais où la détermination des angles est presque toujours absente et en tous cas où leur sens n'est jamais précisé. Nous avons de plus une double médiation pour la réalisation d'une tâche le tracé se faisant par l'intermédiaire des commandes d'un langage qui n'est pas la langue naturelle (LABORDE 1982).

2. Variables et choix des figures.— La séquence a pour objectif de favoriser la construction chez l'élève des connaissances nécessaires pour:

* reconnaître la présence ou l'absence de centre de symétrie dans une figure;

* construire la figure symétrique d'une figure.

Il paraît raisonnable de faire l'hypothèse que pour acquérir ces connaissances l'élève doit être capable de différencier les trois termes qui sont mis en relation : la figure objet, le centre de symétrie et la figure image. Cela nous a conduit à choisir des tâches de tracé de symétriques de figures.

Les variables de la situation sont de deux types :

* les variables des tâches qui concernent la forme des figures et l'emplacement du centre de symétrie et de la tortue,

* les variables de modalités qui concernent le dispositif déjà choisi et l'organisation de la classe.

a) variables des tâches et choix des figures

Trois variables principales apparaissent: la variable figure liée aux seules propiétés de la figure , la variable position du centre de symétrie, la variable position initiale de la tortue. Les figures choisies sont des figures fermées constituées de segments juxtaposés comme pour la symétrie orthogonale. La variable figure nous semble donc caractérisée par le nombre de segments et la variable angle droit qui prend la valeur vrai que si la figure ne comporte que des angles droits.

La variable position du centre de symétrie se partage en trois sous variables suivant que celui-ci appartient au contour de la figure, lui est intérieur ou lui est extérieur. Dans le cas où le centre de symétrie appartient au contour nous avons une sous variable sommet qui prend la valeur vrai si le centre de symétrie est placé en un sommet de la figure et la sous variable milieu d'un segment qui prend la valeur vrai si le centre est au milieu d'un segment de la figure.

La variable tortue se sépare en quatre sous variables qui sont les suivantes:

la variable position du centre de symétrie,

la variable position d'un sommet de la figure,

la variable position du milieu d'un segment de la figuire,

la variable *parallèle* ou *perpendiculaire* à un segment de la figure qui prend la valeur vrai si la direction de la tortue est initialement parallèle ou perpendiculaire à un segment de la figure.

Les figures sont choisies de façon à ce que chaque groupe de variables liées apparaisse avec des choix de valeurs différentes à côté des autres. Les tableaux suivants indiquent le choix des figures en fonction des valeurs des variables.

b) choix des consignes.

Le dispositif choisi est l'utilisation d'un micro - ordinateur avec la liste restreinte des commandes LOGO déjà citées. Les élèves travaillent par paires devant le micro - ordinateur. Ils reçoivent des documents sur lesquels les figures et les centres de symétrie sont tracés. Un des élèves de la paire tape les commandes sur le clavier du micro - ordinateur, l'autre les écrit sur une feuille de papier. Quand les élèves pensent avoir terminé le tracé de la figure symétrique ils appellent la correction. A ce moment on dit qu'ils terminent un essai. Si leur tracé ne coincide pas avec la correction ils doivent faire un nouvel essai jusqu'à concurrence de trois essais. Si au troisième essai le tracé est toujours inexact les élèves doivent lire et noter les commandes réalisant le tracé exact puis les taper au clavier.

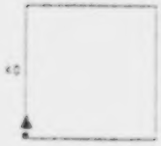
3.Résultats de l'expérimentation et bilan.— L'expérimentation a eu lieu dans trois classes;

- une classe de cinquième (12-13 ans) ayant réalisé la séquence sur la symétrie orthogonale l'année précédente;
- une classe de cinquième ne l'ayant pas réalisé ;
- une classe de sixième venant de réaliser la séquence sur la symétrie orthogonale.

Le CARREE ne comporte pas d'erreur. Le CARREF comporte l'erreur suivante:

	variable position du centre de symétrie					variable tortue				variable figure	
	sur le contour	en un sommet	au milieu d'un segment	à l'intérieur	à l'extérieur	placé au centre de symétrie	en un sommet de la figure	au milieu d un segment	// ou ⊥ à un segment de la figure	nombre de segments	variable angle droit
CARREE	VRAI	VRAI	FAUX	FAUX	FAUX	VRAI	VRAI	FAUX	VRAI	4	VRAI
CARREF	VRAI	FAUX	FAUX	FAUX	FAUX	VRAI	FAUX	FAUX	VRAI	4	VRAI
CARREG	FAUX	FAUX	FAUX	VRAI	FAUX	FAUX	VRAI	FAUX	VRAI	4	VRAI
TRIANGLEC	VRAI	VRAI	FAUX	FAUX	FAUX	FAUX	FAUX	VRAI	VRAI	3	FAUX
TRIANGLED	VRAI	FAUX	VRAI	FAUX	FAUX	VRAI	FAUX	VRAI	VRAI	3	FAUX
TRIANGLEE	FAUX	FAUX	FAUX	FAUX	VRAI	FAUX	FAUX	VRAI	VRAI	3	FAUX
TRIANGLEF	FAUX	FAUX	FAUX	FAUX	VRAI	FAUX	FAUX	FAUX	VRAI	3	FAUX
TRIANGLEG	FAUX	FAUX	FAUX	VRAI	FAUX	FAUX	VRAI	FAUX	VRAI	3	FAUX
MAISONF	VRAI	VRAI	FAUX	FAUX	FAUX	VRAI	VRAI	FAUX	VRAI	6	FAUX
MAISONG	FAUX	FAUX	FAUX	FAUX	VRAI	VRAI	FAUX	FAUX	VRAI	6	FAUX
MAISONH	FAUX	FAUX	FAUX	VRAI	FAUX	FAUX	VRAI	FAUX	FAUX	6	FAUX

CARREE



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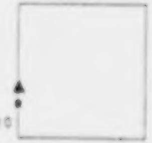
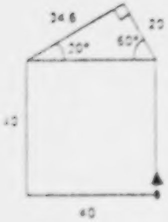


MAISONF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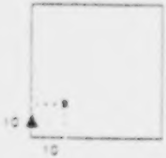
CARREG



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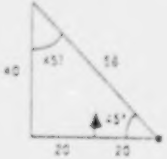
40
45°
56
45°
20
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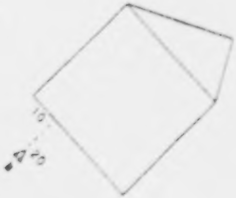
MAISONG



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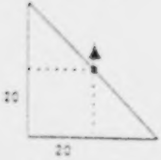

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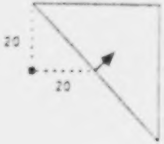

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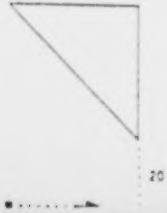

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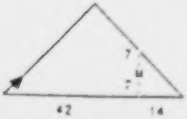
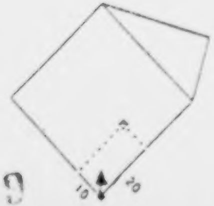
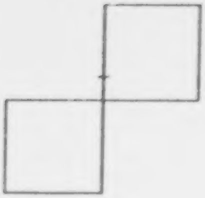
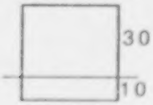




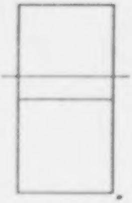
MAISONH



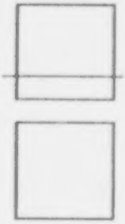
Cette erreur est analogue aux erreurs réalisées pour la symétrie orthogonale pour CARREC qui sont les suivantes:



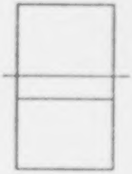
CARREC



Erreur 1

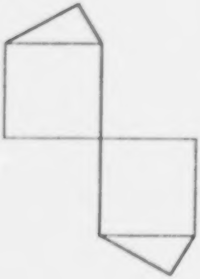


Erreur 2



Erreur 3

Pour MAISONF on trouve comme erreur :



qui correspond à l'erreur d'orientation. On remarque que pour MAISONH cette erreur ne se retrouve que faiblement ce qui indique la présence d'un apprentissage. La classe où on trouve le plus d'erreurs en début de séquence est la classe de cinquième n'ayant pas réalisé la séquence sur la symétrie orthogonale précédemment. Les deux autres classes ont des résultats assez semblables avec cependant plus de rapidité et légèrement moins d'erreurs dans la classe de cinquième ayant réalisé la séquence sur la symétrie orthogonale l'année précédente. En fin de séquence les réalisations des élèves des trois classes se rapprochent et les évolutions au cours de la séquence sont de même type en laissant cependant subsister une différence sensible entre la classe n'ayant pas réalisé la séquence sur la symétrie orthogonale et les deux autres.

Cela nous montre l'importance de l'établissement par l'élève de mises en relation des deux notions: la symétrie orthogonale et symétrie centrale.

4. Conclusion.— Des résultats similaires sont trouvés pour le déroulement de la séquence dans le cas de la symétrie centrale et dans le cas de la symétrie orthogonale. Nous notons cependant que la meilleure maitrise des notions a lieu chez les élèves ayant effectué la séquence sur la symétrie orthogonale en classe de sixième puis celle sur la symétrie centrale en classe de cinquième .

Le dispositif choisi avec le type de procédure de tracé qu'il induit apparaît comme un outil didactique favorisant l'apprentissage des notions de géométrie où intervient l'orientation.

Bibliographie.

GALLOU-DUMIEL E. (1988) *symétrie orthogonale et micro - ordinateur* . Recherche en didactique des mathématiques vol. 8 (1-2 p. 5-60)

LABORDE C. (1982) *Langue naturelle et écriture symbolique deux codes d'interaction dans l'enseignement des mathématiques.* Thèse d'Etat,IMAG,Université GRENOBLE I.

LABORDE C. (1985) *Quelques problèmes d'enseignement de la géométrie dans la scolarité obligatoire.* For the learning of Mathematics 5,3 FLM, Publishing Association Montreal, QUEBEC,CANADA ,pp.27-33.

ABOUT CONTINUOUS OPERATOR SUBCONSTRUCT IN RATIONAL NUMBERS

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We found the existence of a special subconstruct with frational numbers in continuous situation related to Kieren's studies after adding problems to his rational thinking test. Factorial analysis confirm the above researches with Spanish people, and shows different factors with strechers and shrinkers than discrete operator problems.

INTRODUCTION

It's well known the discussion about subconstructs in rational number concepts during the last years. The Kieren's general point of view about intuitional knowledge (Kieren 1988) accepts four central aspects : quotient, measure, ratio and operator, and many relations between them (Kieren 1987). Successive modifications of the rational thinking test (1980-1988) according to this theoretical approach (Rahim-Kieren 1988).

It seem to be similar to the Vergnaud's and Freudenthal's ideas (Vergnaud 1983, Freudenthal 1983) and also for the Rational Number Project (RNP Behr et. al 1983) seem to talk the same ideas with little differences. For instance, Freudenthal talks about fracturer and ratio operator,or transformer (op.cit. pg. 148-149). RNP provides the discrete, continuous and countable situations (perceptual variability) for the fraction concepts (op.cit. 101).

OBJECTIVES

Our purposes were to find if some aspects of variability (Dienes 1971, Behr et al 1983-85) are as different as it seems in a operator context of rational numbers with graphical presentations. It's a part of wide study about knowledge of fractions.

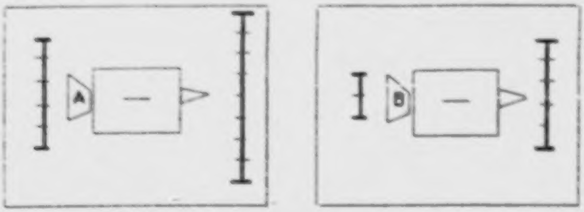
Questions 1) It is possible to have a distinction between the continuous and discrete situations of an operator subconstruct ? 2) Can we accept these operator situations as different subconstructs related the Kieren' schemes ?

METHODOLOGY OF RESEARCH

We used the Rational Thinking Test (version Kieren 1981) In this test there are four parts with 24 general items : ratio situations (mixing xocolat; q.1-6) ,quotient problems (divided pizzas;q 7-14) ,operator discrete problems (input little bars ,ouput boxes;q 15-18) , measure (sharing and drawing surfaces;q 19-24)

We decided to add six more questions about continuous operators bringing the enlargement idea of stretchers and shrinkers (Dienes 1967,Braunfeld 1975,Streefland 1983) to test the possible differences We call these items continuous operator (q 25-30) The items have the same order of difficulties than the used for cognitive levels (Brindley 1980, Noelting 1982) We ask for order comparison between machine situations .

30. Les màquines A i B que veus abaix, allarguen el bastó d'entrada i el fan més gran .



Encercla quina de les dues màquines allarga més ... A igual B

¿Per què ?

Fig Model of item situations

We also add 3 items about continuous measure situations (30-33) bringing ideas from Ratsimba-Brousseau (1981) and Filloy ,Figueras et al. (1987)

We administered the tests to 1800 pupils (grade 5 and 8 ,that are final courses of 2 last periods in Primary Education in Spain). There are 33 items in 3 hours divided by two parts . This test was administered before holidays, and represent the acquisition of concepts at the end of scholarity in each grade.

In each item we consider it's wrong if there is no a convincent explanation of the answer in each case . We assess by exploratory studies (Gimenez 1987) the validity of items.

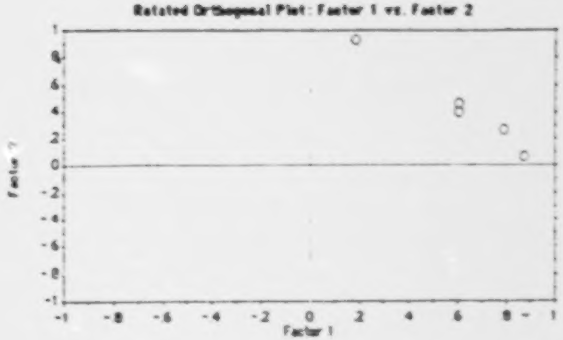
RESULTS

Here we have the significative results of factorial analysis by principal components. We present the rotated orthonormal varimax solution (8th grade for values x ≥ .35)

Items	Fact.1	Fact.2	Fact.3	Fact.4	Fact.5	Fact.6	Fact.7	Fact.8
1		.47						
2		I						
3		I						
4			I					
5			I					
6			I					
7								I
8								I
9		.39						.34
10					I			
11					.44			
12					I			
13					I			
14					I			
15						I		
16						I		
17						I		
18						I		
19							.36	
20							.48	
21							I	
22							I	
23							I	
24								.45
25	.38							
26	I							
27	I							
28	I							
29	I							
30	I							
31				I				
32				I				
33				I				

The 8 factors above presented explain 98% of variance and the single contributions of first to seventh are greater than 10%. The contribution of 8th is 8.3%.

With these results, we assigned to each children an addition of items corresponding each subtest. Then we did the factor analysis of the results of the subtest 1 to 5 (ratio, quotient, discrete operator, measure, continuous operator), by the same procedure, and we can see here the plot of the new factors.



Factor 1	Factor 2
665	-.469
.453	-.177
.15	.197
-.549	1.06
2	.112

We found analogous results with 5th grade students , but really lower level. The only important difference is to present a 9th factor (q. 23.25.26).

Some other results not commented here showed that difficulty is related with item-levels studied by different authors (Noelting 1982, Kieren 1983).

SUMMARY DISCUSSION

The table above presented shows that the items 25-30 ,and 31-33 identify different factors from the Kieren's test and it seems to be an evidence of the perceptual variability. We can also reflect the goodness of RTT (Rahim-Kieren 1988) and the completed items (Gimenez 1988 b).

The different factors 6th and 8th seems to be "sharing" and "fractions as quantities" respectively. The items 1-3 and 4-6 are according to direct or non direct observations on ratio situations.

The plot-design factor-sum explained above shows an acumulation of factors except the 4th. It seems to show the measure conceptual items belong to different categories while the others are more related between them. All this results ask for a new synthetic scheme of literature about fractions (Gimenez 1988a)

REFERENCES

BEHR ,M. et al.(1983) Rational number concepts in Lesh-Landau (eds) *Acquisition of mathematical concepts and processes*. Academic Press . New York.

BEHR,M et al (1985) Main results about Rational Number Project in L.Streefland (ed) *Proceedings of the IX PME* Utrecht. Holland.

BRAUNFELD,P et al. (1976) *Stretchers and shrinkers. Book 3 The theory of fractions*. Harper & Row Publishers . New York.

BRINDLEY et. al.(1980) *Calgary Junior High School Project . Summary Report*. Edmonton.

FILLOY-FIGUERAS-ROJANO *Algunos significados asignados por los ninos al modelo egipcio, fraccion de la unidad*. CINVESTAV del IPN. Mexico.

FREUDENTHAL,H (1983) *Didactical phaenomenology of mathematical structures*. Reidel Dordrecht.

DIENES,ZP (1967) *Fractions* . OCDL .Paris.

DIENES,Z.P. (1971) *Building up mathematics* . Revised ed. London

GIMENEZ,J (1987) *Estudis exploratoris sobre concepcions de fraccions* . Dept. Educació i Psicologia. Unpublished paper . Div. VII. Univ.Barcelona.

GIMENEZ ,J (1988 a) Propuesta metodológica para la ensenanza de las fracciones en la educación básica . Paper presented to the *Reunion Centroamericana de ensenanza de matematicas* S.Carlos de Guatemala.

KIEREN,T-SOUTHWELL (1979) Rational numbers as operators. The development of this construct in children and adolescents. *Alberta Journal of educational research* ,1979 25(4) 234-247.

KIEREN,T(1986) Personal knowledge of rational numbers Its intuitive and formal development in Hiebert-Behr (eds) *Number Concepts and operating in the middle grades* Reston. NCTM , LEA

KIEREN,T(1988) Personal knowledge of rational numbers .It' s intuitive and formal development. in Behr&Hiebert (eds) *Research agenda Project. Research on number learning in middle school*. NCTM. Reston. VA.

KIEREN,T-RAHIM (1988) A preliminary report on the reliability and factorial validity of the rational number thinking test in the Republic of Trinidad and Tobago. in Behr (ed) *Proceedings of the PME-NA*,114-119. De Kalb. Illinois.

RATSIMBA,H. (1981) *Etude de deux méthodes de mésures rationelles "la commensuration et le fractionement de l/unité en vue d' elaboration de situations didactiques*. These de 3eme cycle Univ. Bordeaux.

NOELTING,G. (1982) *Le developpment cognitif et le méachanisme d'equilibration*. Ed. Morin Québec.

STREEFLAND ,L(1983) *Aanzet tot een nieuwe breukendidactiek volgens Wiskobas* . Utrecht. OW&OC .

VERGNAUD,G.(1983) Multiplicative structures in Lesh-Landau (eds) *Acquisition of mathematical concepts and processes*. Academic Press . New York.

Constructivist Epistemology and Discovery Learning in Mathematics

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Many who adopt constructivist approaches to mathematics education base them on radical constructivist epistemology; but as a foundation for research this leads to problems. Empiricism, encompassing nonmechanistic models for knowledge "construction" as well as procedural models for knowledge "transcription," allows equally well the advocacy of constructive mathematics learning through discovery. Some invalid inferences sometimes drawn from radical constructivist epistemology are identified and discussed.

What is the best way to characterize mathematical knowledge and to study mathematics learning? What classroom activities facilitate meaningful learning, and how can teachers be enabled to foster them? One set of research perspectives with which I generally concur includes the following main ideas: **1** Mathematics is invented or *constructed* by people, not an abstract body of "truths" or necessary rules. **2** Mathematical meaning is not transmitted by teachers, but constructed by learners. **3** Guided discovery, meaningful application, and problem solving are more effective than imitation or rote algorithmic symbol-manipulation. **4** Mathematics learning is better observed and assessed through qualitative case studies and individual interviews, not just quantitatively scored skills tests. **5** Effective mathematics teaching does not focus exclusively on correct responses, but encourages diverse, nonmechanical problem-solving processes. **6** Teacher development should include reflections on epistemology--from the historical origins of mathematics to knowledge construction by individuals. As attention has been given to epistemology in the psychology of mathematics learning, the philosophical perspective of radical constructivism has emerged as a justification for views such as these (Cobb, 1981; Confrey, 1986; von Glasersfeld, 1984; 1987; Steffe *et al.*, 1983). Radical constructivists have made important contributions by challenging the premature conclusions and overgeneralizations sometimes drawn from "scientific" research, pointing out that in experimental studies surface variables are often studied because they are easier to make quantitative, while more difficult cognitive variables are disregarded. They have also sought alternatives to the overly mechanical models sometimes offered by the artifical intelligence/cognitive science school. Nevertheless this paper raises some issues in criticism of radical constructivism, arguing for an empiricist approach compatible with the views above that avoids some of its potentially damaging consequences (Kilpatrick, 1987).

Epistemological Perspectives Influencing Mathematics Education

Epistemology is the branch of philosophy that examines the underpinnings of knowledge: what it is, how one acquires it, and the logical (or psychological) bases for ascribing "truth" or validity to it. Much epistemological reasoning begins by analyzing the sources of what "I" (the reasoning entity) know. One such source consists of directly accessible sensory experience (sense-data); another is logical reasoning. Some questions with which epistemologists grapple are: Can I validly infer the existence of external reality *apart* from my own experience? If so how? What can I know about the "real world," and how can I know it? Can I validly infer the existence of other people's internal experiences? If so how? What comparisons can be made between their sense-data and mine? Can I consistently verify the validity of my own reasoning? Are logical and mathematical reasoning intrinsically valid, or only systems of human linguistic convention? What does it mean to say that logical/mathematical statements (seeming to depend on reason) are true, compared to statements in science (seeming to depend on empirical observation)? Is either "truth" objective? Does psychology differ in this from physical or biological science, because its domain is "the mind?"

Answers to such questions have been proposed by exponents of various epistemological schools (e.g. Turner, 1967). **Idealism** is the view that all reality is mental, and no physical real world can be validly inferred. But broadly construed, it may allow the existence of many minds, or even a universal mind with which individual minds share experience. **Solipsism** is the more radical view that the only reality is in *my* mind. At the other end of the spectrum, **causal realism** asserts that an external world exists and *causes* my sense experiences. This view falls within the more general framework of **rationalism** as it asserts that one can acquire knowledge about the physical world *via* reason and logical inference. To the rationalist, sense-data are untrustworthy: they are not the most fundamental reality and may be illusory; one must reason one's way past them to arrive at knowledge of the external world. **Empiricism** relies more heavily on sense-data as elements of a world-as-experienced. Observation and measurement are fundamental empiricist tools, and the predictive value of inferences drawn from patterns in sense-data account for the validity of knowledge. Many epistemologists distinguish *analytic* from *synthetic* truths, though they do not alway agree on their definitions. Roughly speaking, analytic statements (e.g. "All brothers are siblings") are true by virtue of the meanings of the terms involved and cannot be empirically refuted, while synthetic statements (e.g. "George Polya was a mathematician") depend for their truth on empirical evidence. *Reasoning* may nevertheless be required to verify analytic statements; in one view mathematics consists of a body of analytic knowledge. **Logical positivism** is a form of radical empiricism which adopts the "verifiability criterion" of meaning, that the only meaningful content of a synthetic statement lies in the operational methods for verifying it in principle (Ayer, 1946). **Radical constructivism** argues that we can never have access to a world of reality, only to what we ourselves construct from experience; all knowledge (mathematical or not) is *necessarily* constructed. Without telepathic perception one has no direct knowledge of anyone else's world of experience, and can only construct personal models of the knowledge and experience of others. Thus one can never conclude that one's own knowledge is "the same as" another's. Likewise one can only *model* reality, and never conclude that one's knowledge is actually of the real world. In this view knowledge is never communicated, but of epistemological necessity constructed (and reconstructed) by *unique individuals*. Verbal learning entails knowledge construction from in-context experiences of discourse; thus social conventions and interactions rather than "objectivity" often function as the most important determinants of whether a mathematical or scientific concept to be taught has been "correctly" learned. At times epistemology has influenced the psychology and practice of mathematics education, but its implications have not always been correctly drawn. First we discuss logical positivism as a foundation for radical behaviorist psychology and "behavioral objectives" in education (Skinner, 1953; Mager, 1962; Sund & Picard, 1972). Then we examine some aspects of the radical constructivist influence.

The idea of mental states knowable through direct experience is compatible with idealism: since all reality is mental, *behavior* (or, in a more precisely idealist characterization, mental experiences classified as behavior), and *mind* (the full set of mental experiences that I or other human beings have) are on the same epistemological footing. Alternatively, causal realists can posit the *reality* of mental states, treating them as part of a world "out there," knowable in principle by reasoning from their effects on observers. Thus mentalistic explanations of behavior and characterizations of learning outcomes are reconcilable with either idealist or causal realist views. The radical behaviorists, however, rejected mentalistic explanations as *meaningless* (in the sense of logical positivism), involving in-principle-unobservable statements. Their exclusive focus on stimuli, responses, and stimulus-response (S-R) relationships derived explicitly from the fact that these are directly *observable* and *measurable* while presumed cognitive (mental) states are not. It was argued epistemologically that the latter should be excluded *a priori* from scientific psychology--but S-R models never succeeded well in describing insightful mathematics learning. Likewise for instructional objectives to satisfy the verifiability criterion observable, measurable, and thus behavioral learning outcomes must be set in advance. But strict behavioral objectives in mathematics fostered teaching discrete, disconnected rules over developing meaningful patterns or insights. Directly testable goals led to the "efficient" procedure of teaching behaviors directly; accuracy on standardized tests came to dominate instruction, so that teachers now assert with near-unanimity that they have no classroom time for mathematical exploration, discovery, or problem solving. Of course some advocated "back to basics" in mathematics for other reasons: uncomfortable with diversity, they *valued* drill-and-practice or performance-based accountability. But *a priori* principles of epistemology or science do not necessitate these goals.

One need not adopt radical constructivism to pinpoint the error in the positivist epistemological analysis. A moderate empiricist can ask that a meaningful synthetic statement have in-principle-verifiable *implications* without requiring that it be itself directly verifiable. I would not concede that the only meaning of such a statement consists in its presently definable consequences. A model for cognition using unobservable entities (e.g. internal cognitive representations) may succeed in *usefully summarizing and synthesizing observable events* (e.g. behaviors), and may suggest additional observations that were not specified in advance. Such a model is scientific if it gives a more parsimonious description of empirical phenomena than models based on directly observable entities. The early atomic theory in chemistry made use of atoms and molecules then thought to be in principle unobservable; it accounted for certain observations, e.g. those fitting the law of multiple proportions; scientists later found further consequences of the theory, and invented once-unforeseeable ways to observe atoms and molecules directly. The radical behaviorists' *a priori* epistemological reasoning, though wrong, did considerable damage to mathematics education.[1]

Radical constructivism in contrast not only allows but necessitates psychological models for the individual's "understandings" or mental processes. But it has other implications for mathematics education and psychology which should be carefully considered. It's conclusions that all knowledge is constructed and all learning (including mathematics learning) involves constructive processes, are not derived from empirical studies which *distinguish* constructive from non-constructive learning and observe their conditions of occurrence or degrees of effectiveness. Instead they are claimed to follow from *a priori* epistemological considerations: human knowledge is *necessarily* "constructed," from a world of experience. Again as *a priori* epistemological necessities, each person's world of experience (and, therefore, knowledge) is context-dependent: unique and inaccessible to others. Descriptive case studies are not merely a technique in an exploratory stage of empirical study; they are the best that can be achieved, and must *replace* controlled experimentation in mathematics education research because individuals' cognitions are non-comparable. Now the conception of a "mathematical structure" (e.g. the integers and their properties), natural to a

[1] The early behaviorists were reacting at least in part against *particular* mentalistic psychological theories derived largely from introspection, which relied little on systematic empirical observation. Thus they did sweep away much in psychology that was relatively valueless.

mathematician, has been central to "structural" goals in mathematics teaching (Dienes, 1963; Dienes & Jeeves, 1965). The structure of problem representations (search space complexity, etc.) as external to problem solvers is important in task variable research (Goldin, 1984). Radical constructivism denies us these as analytical tools apart from the constructed knowledge of a learner or problem solver, allowing *in principle* no way to establish that a problem or concept "has" the same (or similar) structure for two people: not because of empirical evidence of differences, but due to *a priori* epistemology. Radical constructivist influences on mathematics education have been opposite in direction to that of logical positivism, but both make major claims based on epistemology rather than empirical research. Though sympathetic to the general direction of their influence, I still see dangers if the radical constructivist reasoning is unsound. 1 Advocates of constructive learning *via* discovery processes may find that invalid conclusions from the epistemology are intertwined with otherwise valid perspectives. 2 A valuable and timely set of nonbehavioristic, nonmechanistic ideas in mathematics education may be discredited in the eyes of those who justifiably seek an empirical, scientific basis for research--indeed, recent scholarly debate on issues affecting policy (e.g. variables associated with effective teaching) has been cast as differences between quantitative empiricism and constructivism (Brophy, 1986; Confrey, 1986). Thus I stress that one need not accept radical constructivism to model learning constructively, or to advocate increased classroom emphasis on guided discovery in mathematics.[2]

Constructive and Nonconstructive Empricial Learning Models

Before returning to the epistemological issue let us consider the difference between constructive and nonconstructive *empirical* models for learning. To do so, we define "learning" as the *acquisition by a system or entity of a set of in-principle-observable competencies or capabilities*. We contrast two situations where, because the systems involved are not human minds, questions of epistemology can be deferred: 1 A computer is programmed in a high-level language, e.g. BASIC. Users (even if familiar with the machine's circuitry) need a *helpful model* for its competency acquisition, which detailed knowledge of the electronics is not. Therefore we imagine that the computer represents internally, *via* literal "transcription," the procedures and contingencies that in the program input are expressed in a conventional notational system; and that these are precisely followed in executing the program. The description does not include *how* representation occurs as the program is entered; but there is a useful sense in which no new, important internal systems are built. The "learning" is non-constructive: new competencies are *limited* to processes fully described by the program itself. 2 The body acquires immunity by inoculation, "learning" *via* infection with a killed or weakened virus to defend itself. This biological process is *not* adequately modeled as representation of explicit instructions; scientists may not even know the procedures the body will acquire. It is more useful to conjecture that in interacting with the vaccine material the immune system *constructs* new capabilities, e.g. to recognize the dangerous virus biochemically, or to manufacture antibodies in quantity more rapidly, which are complex and not fully understood. The evidence for a constructive model is obtained *via* controlled, empirical research. Such examples illustrate empirical grounds for distinguishing "constructive" from "nonconstructive" learning: either may occur, in various situations. The hypothesis that immunization (but not programming) elicits constructive processes is testable, and depends not at all on radical constructivist epistemology--immunology and computer science are not helped by saying that "the immune system (computer) has access only to its personal world of experience, not directly knowable by any other immune system (computer), or scientist." Likewise modeling mathematical competence acquisition constructively as part of an empirical theory, or hypothesizing that knowledge construction occurs (e.g., in stages)--is *logically independent* of radical constructivist epistemology. Some learning may be constructive, some not, and the two *empirically* distinguishable. Perhaps constructed knowledge is more widely generalizable and retained longer. The moderate empiricist can define, study, and advocate discovery processes and open-ended problem solving in mathematics, and take account of contextual influences and individual differences.

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[2] "... in some [constructivist] writings the implication seems to be drawn that certain teaching practices and views about instruction presuppose a constructivist view of knowledge. That implication is false." (Kilpatrick, 1987, pp. 11-12)

Constructivist and Empiricist Views of Knowledge

To make explicit the disagreement between radical constructivism and the moderate empiricist position taken here, we return to the question of how "I" (the reasoning entity) acquire knowledge. The (valid) constructivist statement that I have direct access only to my world of experience differs from the (invalid) phrasing that *we* have direct access only to *our* worlds of experience, which tacitly places "me" on the same epistemological footing as other human minds (but presumably not on the same footing as computers or immune systems). It is valid to say that I construct (in an epistemological sense) my "knowledge." Doing so I infer (tacitly, and later overtly) a "real world" to which I relate words and symbols drawn from experience. I then reason about it: as an empiricist I consider my statements as useful summaries of patterns in sense-data, both actual and contingent. I infer in the real world entities called "other people," and in another epistemological step reason that they too have "worlds of experience." This organizes *my* experience of their behavior: people *seem* to act *as if* they feel sensations and have thoughts like my own.

Thus in modeling their cognitions (e.g. to teach mathematics) I begin with my *own* experiences and infer a description of *their* knowledge: informally, or with systematic empirical techniques such as controlled experimentation. Others' behavior and cognitions are for me on the same epistemological footing as any other aspect of the real world (such as atoms and molecules).

The fact that I wish to study cognition rather than chemistry has slight effect on the epistemological underpinnings of my investigative methods. In cognitive (unlike chemical) studies, it may help to establish and reason from similarities between others' behavior and mine, and correspondences between my behavior and subjective experiences. But such techniques have limitations: it is apparent that other people differ from me behaviorally in important ways, and there is empirical evidence that my awareness and recollection of my behavior and subjective experiences are imperfect. Thus reasoning about others' cognitions by analogy with my own is only a heuristic tool--it may guide some theorizing and motivate some everyday teaching activity, but it must yield to more rigorous empirical investigation when the latter is possible. Thus I argue that it is epistemologically invalid to take as equivalent (a) the "knowledge" of others that I or other researchers model when we study cognition empirically, and (b) the inner "knowledge" that I construct from my personal experience. *These two senses of "knowledge" differ*: one is a defined, *useful shared construct* enabling researchers and teachers to better predict or influence behavior; the other is accessible only to introspection. Whether empirically-defined knowledge "really" describes inner knowledge-as-constructed is not an issue, because *that is not its intent.*

Epistemology and the Psychology of Mathematics Education

Mathematics seen *logically* is a set of assumed conventions for manipulating symbols. Once these are established, there is a sense (contrary to radical constructivism) in which the system *exists* and "has" a structure, apart from the individual. Historically conventions were *invented*, and psychologically they are reinvented by individuals; but their *consequences* are logically constrained. Though a logical formalism is useful, we stress its *empirical* motivation: e.g. the commutative law of addition, *assumed* in one formal approach to number theory, can be *discovered* if we first interpret addition as a physical procedure. To talk about discovering a pattern or structure we must view it as existing apart from the individual. One guides a child to "invent" counting, or "invent" addition by joining sets of objects; having done so the commutative law is not invented but discovered: it exists in that context apart from the child's cognition. To guide the discovery a teacher must know of its existence, and foster situations in which it can be found and interpreted. To encourage *meaningful* over *rote* mathematics learning, we must distinguish them empirically.

One distinction focuses on teaching and learning strategies, which may range from the teacher stating and exemplifying rules, to the student detecting patterns and verifying conjectures. Another focuses on the empirically observed capabilities of students who have "learned" an arithmetic rule: stating the rule, applying it to numerical examples when asked or spontaneously, identifying presented instances of it, providing exemplars and non-exemplars, are important but can be acquired *via* rote, fairly nonconstructive procedures. Other capabilities, e.g. illustrating the rule physically, justifying it, or setting up a pattern where it can be discovered, suggest more meaningful learning. The latter go beyond computation to connect numerical with non-numerical domains, or make explicit reference to reasoning processes. I think it is empirical fact, not epistemological necessity, that methods based on "transcription" and application of rules are less successful than those of mathematical discovery and constructive learning. Criticism of radical constructivism is *not* support for behaviorism or rule-governed learning in mathematics, but a call for new empirical models encompassing far more complex capabilities (Goldin, 1987).

References

Ayer, A. J. (1946) *Language, Truth, and Logic* (2d ed.), NY: Dover.

Brophy, J. (1986) Jour. for Res. in Math. Ed. [JRME] **17**, 323-346; *ibid.* 361-368.

Cobb, P. (1981) in C. Comiti & G. Vergnaud (eds.), *Procs. of the 5th int'l. conf. of PME*, Vol. 1, Grenoble, 50-55.

Confrey, J. (1986) JRME **17**, 347-360.

Dienes, Z. (1963), *An experimental study of mathematics learning*. London: Hutchinson.

Dienes, Z. & Jeeves, M. (1965), *Thinking in structures*. London: Hutchinson Educational.

Goldin, G. (1984) in G. Goldin & C. McClintock (eds.), *Task variables in mathematical problem solving*. Phila.: Franklin Institute Press [now Hillsdale, NJ: Erlbaum], 103-169.

Goldin, G. (1987) in C. Janvier (ed.), *Problems of representation in the teaching and learning of mathematics*. Hillsdale, NJ: Erlbaum, 125-145.

Kilpatrick, J. (1987) in J. Bergeron, N. Herscovics, & C. Kieran (eds.), *Procs. of the 11th int'l. conf. of PME*, Vol. 1, Montreal, 3-27.

Mager, R. (1962) *Preparing instructional objectives*. Palo Alto, CA: Fearon.

Skinner, B. F. (1953) *Science and human behavior*. NY: The Free Press.

Steffe, L., von Glasersfeld, E., Richards, J., & Cobb, P. (1983) *Children's counting types: Philosophy, theory, and application*. NY: Praeger.

Sund, R. and Picard, A. (1972) *Behavioral objectives and evaluational measures: Science and mathematics*. Columbus, OH: Merrill.

Turner, M. (1967) *Philosophy and the Science of Behavior*. NY: Appleton-Century-Crofts.

von Glasersfeld, E. (1984) in P. Watzlawick (ed.), *The invented reality*. NY: Norton, 17-40.

von Glasersfeld, E. (1987) in C. Janvier (ed.), *Problems of representation in the teaching and learning of mathematics*. Hillsdale, NJ: Erlbaum, 3-17 and 215-225.

COMPARATIVE ANALYSIS OF TWO ARITHMETIC SITUATIONS IN SEVEN YEAR OLD CHILDREN

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Summary

A sample of 105 seven-year-old children were subjected to a test consisting of two semantically similar arithmetic situations, but different in the amount of information and structure. The comparison of the responses given by the subjects in the two situations indicated that the modalities of response varied from situation to situation, even for the same child. Whoever, in the first situation, had used a consolidated and internalised subtractive procedure, in the second used other modalities which again return to graphic representations and a restructuring of the information.

Introduction

The aim of the present work is to study the strategies used by a group of children, aged seven, in the solution of two arithmetic situations; addition and subtraction.

The comparative analysis between the two situations has the intent of pointing out, within the theoretical framework of cognitive psycology, how the additive - subtractive procedures are acquired by children who are at the stage of concrete operations. Such research is done within the authorised activities conducted by the Didactic Research Group (Nucleo di Ricerca Didattica) of the University of Pavia which proposes to put into effect, in the elementary schools, a mathematics curriculum which emphasises the critical-formative aspect of the disciplines. In this regard, the study of the cognitive strategies used by the children gain particular importance.

Here we will limit ourselves to the presentation of a strong point of the investigation that we are conducting on the verbal additive situations.

Methodology

The research was conducted on 105 seven-year-old subjects attending the second elementary class.

Our children belonged to six different schools, representing the social and economic classes of the population of the city and province of Pavia.

The trials were composed of two arithmetic situations which the children had to solve on successive days to avoid the possibility that choice of solution of the first could, according to the logic of the "fixity of the task", influence the solution of the second.

The two arithmetic situations were undertaken by the subjects after about three months from the beginning of the second year of elementary school when addition and subtraction were presented in the cultural baggage of the students.
The two situations proposed are the following:

A. There are 18 gifts under the tree on Christmas morning:
Three children enter and take their gifts from under the tree;
6 gifts remain for their parents;
How many gifts have the children taken?

B. This morning the baker had 30 buns;
He sold 4 of them to Roberta, 2 to Luca, and some to Mario;
Now he has 18 left;
How many did he sell to Mario?
How many did he sell altogether?

If we use Moser's (1985) semantic classification of the proposed problems, our situations can both be placed in the class Transforming/Separating.

The semantic classification of the problems begins from the situations of action or staticity which the problem describes for identifying the logical operators like union and/or separation in situations of transformation, combination, comparison and equality.

The second problem proposed by us presents, however, a syntactic and organisational structure of the information which is different with respect to the first. In fact, the order of the questions is not one which conforms to the sequentiality of the actions described. The text also contains a problem of combination of the subtractive type. Further, to put into play the relations which exist between a particular set and three of its subsets does not indicate any implicit actions and, therefore, any reference to the strategies of separation and addition, but it describes the relation which occurs between the quantities.

Our problem was that of seeing how the same subjects behaved in the two situations. Thus, the different subtractive strategies used to resolve the trials were pointed out.

For the first situation, three strategies were used.

I) 6 + = 18 In this case the modality used by the child is additive, he begins from the smallest quantity and builds a larger one by the addition of objects until reaching the greater number. The counting of the objects added gives the result.

II) 18 - = 6 In this case the modality used is that of "separating up to". The child removes, from the more numerous set, many units up to the point when the number remaining equals that of the smaller given.

III) 18 - 6 = The modality used here is that of "separating from". The child takes away from the larger of the given sets the number of units indicated by the smaller set.

These three subtractive strategies are also represented in the second situation, even if combined among themselves in a different way. There are, in fact, four modalities which we have compared.

I) 30 - 4 - 2 - = 18 In this case the child used the modality "separating up to", using more sets and then those of the sums between two quantities which were not directly given by the text, but which had to be found by the child.

II) 4 + 2 + 18 =

Here the child uses an additive modality to build the subset of the objects that he knows, then using the modality "separating from" to find the unknowns of the problem.

III) 30 - 18 =

In this case the subtractive modality "separating from" is also used.

IV) 30 - 4 - 2 - 18 =
6 + 6 =

In this last case the child unites to the modality "separating from" that of the sum between two quantities.

It is interesting to note, above all in the second situation, how the children demonstrate a "flexibility" not so much in the operative modality as in the organisation of the information that they have at their disposition. In using the same subtractive strategies, the data are used in a different way.

Results

Of our 105 subjects, 68% (71 students) responded correctly to both trials, 25% got only one trial wrong (8.5% the first and 16.5% the second) while only 7% got both trials wrong.

Now let's study the modalities used in the two different situations; examining the correct protocols.

Situation A	Correct 83.8%	Situation B	Correct 77.13%
Modality		Modality	
I	12.38%	I	38.09%
II	20.00%	II	14.28%
III	51.42%	III	8.57%
	83.80%	IV	16.19%
			77.13%

Of our subjects, 83.8% responded correctly to the first situation and 77.13% to the second.

In situation A the most used modality (51.42%) was that known as "separating from". In this case the information was used in the given order. Only 31% (17 out of 54) of the children doing the third modality also used the graphic representations of the objects. Instead, we observe that 71.4% of the children who used the modality "separating up to" made use of the representations of the objects; representation used by all the subjects that resort to the first resultive modality, the additive one.

Therefore, it seems that the children who resort to the third modality are already able to use an internalised plan of action which does not need the objects to be activated.

In situation B we encounter the highest percentage (38.09) of the use of the strategy "separating up to". However, we must observe that the other three modalities use the strategy "separating from" which therefore, as a whole, was used by 39.04% of the subjects who responded correctly to the second situation. Moreover, it should be noted that the use of the graphic representations was increased. A good 80% of the children who used the first modality made use of the representations, 66.6% of those using the second modality and 82.3% using the fourth, while none of the subjects who used the third modality used them. The comparison of these percentages allows us to hypothesize that the internalisation of the work plan of subtraction does not always follow a linear process, but facing information of a greater quantity and which is more difficult to organise, as in the case of situation B, the children resort to procedures which can still be verified at a perceptive level following the technique of counting.

In the following table the data of the children who correctly solved both the first and the second modality are reported with reference to the modality used.

		SITUATION A			
MODALITY		I	II	III	
SITUATION B	I	5	12	18	35
	II	1	3	9	13
	III	1	1	5	7
	IV	3	//	13	16
		10	16	45	71

Of the 45 subjects who, in the first problem, used the modality "separating from", 18 did not use it in the second, while 27 used it with different modalities. This would seem to confirm the initial observation that the 7-year-old children who used the subtractive strategy "separating from" elaborated an operative model that, in most cases, no longer resorted to the object.

We must, however, recognise that only 5 subjects who used the third strategy in situation B effectively demonstrated that they had internalised the plan of "separating from" , avoiding both the graphic representation of the objects and the sequential use of the information in favour of a process of synthesis.

Moreover, it is interesting to underline the fact that of the 35 subjects who, in situation B made use of the modality "separating up to", 18 had used the modality "separating from" in situation A. These data become clearer by the last analyses of the protocols from which it results that a good 14 of these subjects had, however, used graphic representations in situation A, witnessing to the necessity of still connecting the thought to the object.

Conclusions

The results obtained in this research point out how the internalisation of the subtractive processes are much more complex than they had seemed.

Our subjects, at 7 years of age, having, by all means, understood the meaning of such processes, present some differentiation from the operative point of view.

Two semantically similar arithmetic situations resulted from the same subjects according to different modalities of calculation which vary in relation to the amount of information and, in general, to the context.

Therefore, it would seem that the acquisition of the subtractive procedure is also subjected to a "cognitive flexibility" which can re-enter into the capacity of the creative thought. Lacking a plan internalised in an adequate way, the subject resorts to his own abilities of creative thought trying other solutions.

References

Carpenter, T.P., Moser, J.M., Romberg, T.A.: (1982) Addition and Subtraction: A Cognitive Prospective, Hillsdale, NJ, Erlbaum.

Gelman, R., Gallistel, C.R.: (1978) The Child's Understanding of Number, Cambridge, MA, Harvard University Press.

Mosconi, G., D'Urso, V.: (1973) La soluzione di problemi, Giunti Barbera, Firenze.

Mosconi, G.: (1981) 'All'origine dei problemi: l'incompatibilità presente nella mente.', in Education alla ricerca e trasmissione del sapere, a cura di Tornatore, L., Loescher, Torino.

Mosconi, G.: (1986) 'Studio su due problemi proposti da

Wertheimer', in Giornale Italiano di Psicologia, a.XIII, n.3, settembre.

Moser, J.M.: (1985) 'Analisi delle strategie di risoluzione dei problemi verbali', in Artusi Chini, L., Numeri e operazioni nella scuola di base. Aspetti psicologici e processi cognitivi, Zanichelli, Bologna.

Moser, J.M.: (1985) 'Alcuni aspetti delle più recenti ricerche sull'apprendimento dei concetti e delle abilità fondamentali dell'addizione e della sottrazione', in Artusi Chini, L., Numeri e operazione nella scuola di base. Aspetti psicologici e processi cognitivi, Zanichelli, Bologna.

Passolunghi, M.C.: (1984) 'Modelli cognitivi del calcolo mentale: le addizioni', in Giornale Italiano di Psicologia, a.XI, n.1, aprile.

Piaget, J.: (1972) La formazione del simbolo, La Nuova Italia, Firenze.

Piaget, J.: (1975) La presa di coscienza, Etas Libri, Milano.

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References

Carpenter, T.P., Moser, J.M., Romberg, T.A.: (1982) Addition and Subtraction: A Cognitive Prospective, Hillsdale, NJ, Erlbaum.

Gelman, R., Gallistel, C.R.: (1978) The Child's Understanding of Number, Cambridge, MA, Harvard University Press.

Mosconi, G., D'Urso, V.: (1973) La soluzione di problemi, Giunti Barbera, Firenze.

Mosconi, G.: (1981) 'All'origine dei problemi: l'incompatibilità presente nella mente.', in Education alla ricerca e trasmissione del sapere, a cura di Tornatore, L., Loescher, Torino.

Mosconi, G.: (1986) 'Studio su due problemi proposti da

Wertheimer', in *Giornale Italiano di Psicologia*, a.XIII, n.3, settembre.

Moser, J.M.: (1985) 'Analisi delle strategie di risoluzione dei problemi verbali', in Artusi Chini, L., *Numeri e operazioni nella scuola di base. Aspetti psicologici e processi cognitivi*, Zanichelli, Bologna.

Moser, J.M.: (1985) 'Alcuni aspetti delle più recenti ricerche sull'apprendimento dei concetti e delle abilità fondamentali dell'addizione e della sottrazione', in Artusi Chini, L., *Numeri e operazione nella scuola di base. Aspetti psicologici e processi cognitivi*, Zanichelli, Bologna.

Passolunghi, M.C.: (1984) 'Modelli cognitivi del calcolo mentale: le addizioni', in *Giornale Italiano di Psicologia*, a.XI, n.1, aprile.

Piaget, J.: (1972) *La formazione del simbolo*, La Nuova Italia, Firenze.

Piaget, J.: (1975) *La presa di coscienza*, Etas Libri, Milano.

Understanding and discussing linear functions in situations. A developmental study.[1]

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This study is a first attempt to investigate 7th, 9th and 11th graders' functional reasoning in situations and to understand how and when general expressions are spontaneously used in such a context.
Results indicate that all the groups had good understanding of key features of linear functions in the situation, but that only 11th graders understood that conceptual descriptions may work as answers where lack of information does not allow numerical answers. 7th and 11th graders spontaneously adapted the level of generality of their solutions to the level of constraint of the problems. 9th graders focused more on the way the different variables of the situation interacted than on trying to compute specific answers even in highly constrained problems.

One can accept the first part of the following claim :"Generality is the lifeblood of mathematics" (Routes to/Roots of Algebra. Open University, Centre for Mathematics Education, 1985, p.8) without buying its tail "and algebra is *the* language of generality". (our emphasis). Potentially an ideal way to learn to deal with concepts and numbers together, algebra, as taught in the schools, is now widely charged with "too much meaningless symbol pushing" (Kaput, 1987, p. 345).

Linear functions represent the easiest way to keep expressing generalities while reintroducing meanings. This is not a new idea and Wheeler and Lee (1987) see its origin in the 1920s. Well documented studies have shown (Piaget, 1968) that mastery of the symbolic notational system is not necessary to understand functional relations, and that the use of situations may successfully trigger functional reasoning among students (Janvier, 1978). Most school curricula however delay the study of functions until the introduction of algebra.

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[2] Currently visiting at Stanford University, sponsored by the FNRS Fellowship 81.353.0.86.

In this study, we have tried to capture in the language of 7th, 9th and 11th graders marks of an understanding of functions beyond the simple statement of the relations between the variables of the situation, and to analyse its strengths and weaknesses. We also wanted to take a first look at what makes students spontaneously decide what level of generality is appropriate for a given situation.

The experimental situation

The data come from the protocols of 9 pairs of subjects, three of 7th, three of 9th and three of 11th graders, with respectively, 0, 1 and 2 years of algebra at school. They were asked to answer questions and solve problems related to the functioning of the device shown in Figure 1. Each interview lasted 50 minutes.

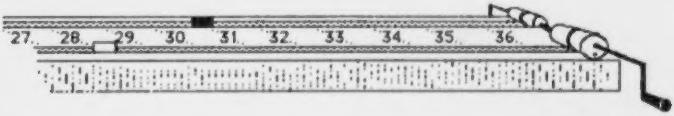

Fig. 1 The device used in the experiment.

Two blocks can be moved up along parallel tracks by turning the wheels situated at one end of the device. The wheels can be actionned independently or together. Wheels of different sizes can be put on each crank, allowing the blocks to move 2, 3, 4 or 6 inches per turn. Different starting positions can also be given to the blocks. Each track may be seen as embodying a linear function of the form $y=ax+b$, in which y represents the ending positions of the blocks, a the sizes of the wheels, x the number of turns and b the starting positions of the blocks. The use of two tracks, aside of proving highly motivating by introducing an element of competition, was decided to allow the discussion of linear functions, beyond one particular instantiation.

The first 15 minutes of the sessions served as an open-ended **familiarisation** phase. Its aim was to let the subjects become acquainted both with the device and with questions designed to promote generalisation by offering only part of the necessary information. "What will happen when you turn the handle?" or "Where will the blocks be when you stop turning?" are example of those initial questions.

For the next fifteen minutes, subjects were given **problems** to solve. These problems presented various level of constraint. The less constrained problems like : "How could you make the blue block be at 20 first?", were solvable in many different ways. More constrained problems, like "Have both blocks be at 24 at the same time?" accepted fewer solutions. In both cases, the experimenter kept prompting for other ways, until evidence was received that the subjects had considered possible action on the three variables involved (i.e number of turns, sizes of the spools and starting points). In the totally constrained problems, enough information was introduced into the situation, (for instance by selecting two spools and two starting positions), that only one solution remained possible.

At the end of the problem phase, another 15 minutes were reserved to ask the subjects to make **Inferences** in order to assess their degree of understanding of some key features of linear functions. As in the totally constrained problems, the questions concerned particular settings of the device. Given the usual difficulties raised by problems about ratios, a central question was the understanding that the ratio of the travelled distances of the two blocks was constant and equivalent to the ratio of the sizes of the two spools. In the simple case where both blocks started at 0 and had respectively a 3 and a 6 spool, we asked successively: "Will the number that this block is at (the one with the larger spool) ever be two times the number that this one is at?" "Will it always be so?" and "Will this number ever be three times the number of the other block?". Argumentations were always requested.

The sessions ended by a presentation of an **abstract situation,** in which the sizes of the spools and the starting positions were only given in a relative way ("Now let's imagine that we have a bigger spool on one side and a smaller one on the other. I won't tell you the size of the spools I have in mind, however. The block pulled by the smaller spool starts ahead, but again I won't tell you how far ahead.) Questions like: "Can you still say anything about when they are going to be on the same line?" were discussed in order to assess the subjects' understanding that general, conceptual answers remained possible when numerical answers were no longer to be found.

Results

Problems : subjects were voluntarily left totally free to decide whether and how much they wanted to use the device in their solutions. The interviewer accepted specific answers -- solutions involving mention or use of at least one particular spool, starting position or number of turns (ex: *Put a 6 on this one*), as well as unspecific solutions -- answers containing only relative descriptions of those parameters (ex : *Have a bigger spool on this one*) and no specific values.

For the less constrained problems, all age groups gave predominantly unspecific, but working solutions, (respectively 60, 74 and 70 % for the 7th, 9th and 11th graders). This result indicates that unspecificity about the values of the variables is seen as the appropriate answer to a weakly constrained problem. For more constrained problems, the proportion of unspecific solutions drops significantly for the 7th and the 11th graders to respectively 31 and 25 % but stays high among the 9th graders (75 %, $p < .05$). This result shows that even the 7th graders are able to adapt the level of generality of their solutions to the level of constraint of the problems. The unspecific solutions given by the 9th graders should not be regarded as weaker performances. They are indeed often well formulated (4A: *Well, I think you have to use the same size spool, or if you used a smaller one you'd have to start it ahead*) and can be followed by very accurate answers on request, as shown in the following interchange.

Interviewer : *All right, suppose you want to make the red end at 24 and the blue end at 8. Can you construct that situation?*

Student 6A : At the same time?

Interviewer : *Yes, at the same time, a tie, or you know...*

Student 6B : If you took, um, a 6-spool, a 2-spool, and like, you did, um, wait, oh.

Interviewer : *The red's gonna be at 24, and the blue's gonna be at 8.*

Student 6A : You can have the 2-spool to the blue, and the 6-spool to the... I mean the 6-spool to the red...and then have the same number of turns.

Student 6B : You can crank that one (red) three times as much as this one.

This tendency of the 9th graders to remain unspecific even in the more constrained problems may be due to a special effort to capture and express how the variables of the situation can be manipulated and interact to produce the target event (like a meeting) rather than to try to pursue one specific solution that would make the target event to happen at the specified location (24). Another finding of this research offers support for this interpretation. Use of comparatives throughout the session (like smaller, faster, etc) is more frequent among the 9th graders than in any other group (appearing in 25 % of their general statements and in only 15 % for the 7th and 11 % for the 11th graders).

Totally constrained problems are generally received with guesses and requests to use paper or to work them on the device. Correct solutions appear in all age groups and are usually found by constructing tables and comparing the positions of the two blocks after each turn. One 11th grader however got the correct answer to the problem : "How many turns will it take for the two blocks to be equal?" by immediately computing the gain per turn of the bigger spool and comparing it to the head start of the other. (7B: *I knew that if this was 4 and this was 3, every turn this would gain 1, and they are 5 apart, so 5 turns would put'em together*), attesting that he also understood that the function resulting from the composition of two linear functions was also linear. Two pairs of 11th graders tried to use equations to verify their correct solution but failed to set the equations correctly.

Inferences : Only one pair of subjects (9th graders) accepted that, when starting both from the 0 mark, blue could eventually later be three times as far as red after having seen it be two times as far at one particular point. All the other pairs (except for one pair of 11th graders for whom the question was considered too obvious) explicitly argued that, with a joint start at the 0 mark, the ratio between the positions of the two blocks would stay constant and equal to the ratio of the spool sizes. One can hardly be more explicit than this 9th grader : *the blue spool is twice as large as the red spool, so it can't move any faster than twice as far*

Subjects' reactions to the **abstract situation** show a clear developmental pattern across levels. In all the 7th grade pairs, subjects proposed to supply the unspecific information given by introducing values of their own. No other pair did so. The four 9th graders dealing with the question considered that because of the unspecific information given about the spool sizes and the starting positions nothing sensible could be said about where the meeting of the two blocks would occur. (5A: *We don't know*, to which 5B added : *It all depends on how much bigger the spool is* or 4A: *You need to know how many inches it moves per spool*). This result is somewhat surprising coming from the very same subjects who had proposed unspecific solutions even for the more constrained problems earlier in the session. Further investigation is needed to better understand this apparent paradox. Answers at the level of the variables were given only at the 11th graders' level who gave answers like: 9A: *The red will catch up with the blue at the point at which it's size advantage can cover the original loss in distance*, or 7A: *the number of turns it takes for blue to catch up with red will depend on the size of the spool*, immediately completed by 7B: *and the head start*.

This result is consistent with another finding of this research showing that if the use of the variables' names rises remarkably at the 9th graders level already, those names remains absent of expressions involving words like *it depends on* or a quantifier like *twice* or *half*. While 9th graders produced almost only sentences like : *it depends on how big the spool is* or *it will go twice as far*, the 11th graders used mainly expressions like *it depends on the size of the spools* (79 % of their "depend" expressions involved concepts' names, a highly significant increase, $p<.001$). Expressions like *it's twice the distance* started to be more than just the exception among the 11th graders only (33 % of their quantified expressions involve the variables's names against only 11 % among younger subjects). "Composite" formulas like : *the smaller size spool* , or *twice as much distance* , while also attesting of the 9th graders effort to integrate concepts into their expressions, reveals that they are not yet ready to have the concepts supplant the comparatives.

Discussion

This study represents a first attempt to understand how and when 7th, 9th and 11th graders form and use general expressions to characterize functions in situations. It was shown that, when reasoning about a situation, even 7th graders had a good enough understanding of linear functions, to correctly answer questions about the ratios of the images of two linear functions with same intercept. Only the 11th graders however introduced the variables' names in expressions involving mention of dependency or mathematical relations, although the use of those names was frequent among the 9th graders, in other types of sentences. Half of the 11th graders understood that expressions at the level of the variables could be given as solutions for problems where lack of specific information made numerical answers impossible.

The 7th and the 11th graders correctly adapted the level of generality of their solutions to the level of constraint of the problem. The 9th graders, though generally able to find specific solutions on request, tended to stay at the same level of generality for the more than for the less constrained problems. In both cases, they often focussed more on expressing the possible ways to realize the desired event (like use a bigger spool, to make a particular block win, or give one a bigger spool and the other one a head start, to get a tie) than on the specific values that would make this event happen at a particular location on the device, as also requested in the problem. Additional data suggest that this tendency is to be seen as reflecting how the 9th graders interpreted the interviewer's expectations in the situation and not as a drop in quality of their reasoning. Interestingly the interviewer even accepted as a good solution for the problem : "How to make the blue block be at 20 first?" an answer as ambiguous as : 5A: *Put a different spool.* This result shows that subjects do make some hypotheses on the level of precision they have to go to and the level of understanding they will be credited for, when they deliver general answers. Very often, this level was determined through negotiation between the students, making the use of pairs of subjects a very appropriate technique where the form of the message is as important as it's content. This also shows that, if generality is the lifeblood of mathematics, more theoretical work has to be done in defining generality for verbal expressions.

References

Janvier, C. (1978). The interpretation of complex Cartesian graphs representing situations: Studies and teaching experiments. Doctoral dissertation, University of Nottingham.

Kaput, J. (1987). PME-XI Algebra Papers: A Representational Framework. In *Proceedings of the 11th International PME Conference*, Montreal, Vol I, 345-354. Open University, Centre for Mathematics Education. *Routes to/Roots of Algebra*. The Open University Press, 1985.

Piaget, J. , Grize, J., Szeminska, A., & Bang, V. (1968). *Psychologie et Epistémologie de la Fonction*. Paris: Presses Universitaires de France.

Wheeler, D. & Lee, L. (1987) *Algebraic Thinking in High School Students: Their Conceptions of Generalisation and Justification*. Department of Mathematics, Concordia University, Montreal.

LOW MATHEMATICS ACHIEVERS' TEST ANXIETY

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Abstract

Reactions To Tests questionnaire (Sarason, 1984) was used to check eighty two low achievers in mathematics in order to find out their level of test anxiety. Counter to theoretical expectations, which predicted a high level of test anxiety, findings show indifference to taking of tests among these students. Implications for curriculum indentation are discussed.

The Problem and Its Background

Test anxiety has been widely studied (e.g. Helmke, 1988; Hembree, 1988; Sarason, 1984). Evidence of a negative correlation between test anxiety and performance in evaluative situations has led to a wide variety of experiments, aimed at evaluating hypotheses about the processes that may be involved. (Sarason, 1988).

We chose to check test anxiety of low ability students in vocational high schools. The system of vocational high schools in Israel runs in parallel to the academic high schools. In vocational high schools students major in technical fields such as mechanics, constructions, secretarial work, etc. In the context of evaluating a newly developed mathematics curriculum for these students, the problem of achievement evaluation arose. Naturally, it became apparent that testing for achievements should be considered as an instrument. In view of the well established history of failures in mathematics examinations of this population, and in accordance with Sarason's findings mentioned above, it was reasonable to expect high level of test anxiety on the part of these students. Therefore we decided to check it. Existance of test anxiety certainly would violate achievement tests' validity. One must, however, be very careful in concluding the opposite if test anxiety measure proves that it does not exist. Namely, if test anxiety is too low, this also may violate the validity of the results.

The Instrument

Test anxiety was measured using Sarason's (1984) Reactions To Tests (RTT) questionnaire. As both general and test anxieties are usually defined as complex states which include cognitive, emotional, behavioral and bodily components, Sarason's instrument consists of four factor analytically derived scales :

T: Tension (e.g. "I feel distressed and uneasy before tests")

W: Worry (e.g. "During tests I wonder how the other people are doing")

IT: Test-Irrelevant Thought (e.g. "Irrelevant bits of information pop into my head during a test")

BS: Bodily Symptoms (e.g. "My heart beats faster when the test begins")

Altogether, the RTT questionnaire consists of 40 statements, each with four alternative reactions. Examinee is to circle the alternative that best reflects how he or she reacts to the statement. The four alternatives are:

1 - not at all typical of me
2 - only somewhat typical of me
3 - quite typical of me
4 - very typical of me

As there are 40 statements in the questionnaire, ten of each scale, a student could get a score of between 10 to 40 for each scale, and a score of between 40 and 160 for the whole questionnaire.

The instrument was validated by Sarason using it for normal students' population in the United States (1984). Michaelis et al. (1988) translated it into Hebrew, and validated the translation applying the test to 54 Tel Aviv University students, who sought counselling because of their suffering from test anxiety.

The Sample

The RTT instrument was administered to 82 low ability vocational high school students at average age sixteen, 45 boys and 37 girls. Their mathematics achievements, as well as other achievements throughout school, were below average, therefore in high school they were assigned to the low ability vocational stream. A study of their characteristics (Movshovitz-Hadar, 1987) revealed a low motivation coupled with a variety of social and learning problems.

Results

The mean questionnaire scores and standard deviations are presented in Table 1. In that Table we also present, for the sake of comparison, results obtained by Michaelis et al. (1988) and by Sarason (1984).

TABLE 1: RTT MEANS AND STANDARD DEVIATIONS

	PRESENT STUDY Low ability vocational high-school students in Israel						MICHAELIS ET AL. STUDY Students in Israel with admitted test anxiety who asked for counselling (1988)						SARASON'S STUDY Normal Students in the United States (1984)					
	MALE N=45		FEMALE N=37		TOTAL N=82		MALE N=16		FEMALE N=38		TOTAL N=54		MALE N=144		FEMALE N=241		TOTAL N=385	
	M	SD	M	SD	M	SD	M	SD	M	SD	M	SD	M	SD	M	SD	M	SD
T	16	4.6	19	5.1	17	5.1	31	5.9	35	5.4	34	5.7	22	6.6	25	7.6	NO	
W	21	5.4	22	4.0	21	4.9	28	5.6	27	5.9	28	5.7	20	5.9	21	6.7	DATA	
IT	15	5.2	16	6.1	15	5.5	22	7.6	22	7.7	22	7.6	17	6.4	18	7.0		
BS	12	3.2	15	4.8	13	4.2	20	5.8	23	5.9	22	5.9	15	4.1	16	5.7		
TOTAL	64	14.9	71	14.9	67	15.4	102	20.3	107	17.5	106	18.4	74	18	80	21		

Table 2 presents distribution of results by scales and sub-groups of scores.

TABLE 2

Anxiety Scores	T	W	IT	BS
Low (10-20)	68	44	68	75
Med (21-30)	12	33	14	7
High (31-40)	2	5	0	0

Discussion

Counter to our theoretically based expectations, results in Table 1 show that students of both sexes in the low mathematics achievements population have a very low test anxiety score, compared to that of students of both sexes of high test anxiety population in Israel, and to that of normal students' population in the United States.

Table 2 shows that hardly any student received a score higher than 30 in each of the four scales. To re-examine this gap between theoretical prediction and our results, we interviewed the mathematics teachers of these students. Teachers rejected the theory and were not at all surprised to hear about their students' low test anxiety. On the contrary, they said it was consistent with their own expectations and observations. According to their experience, they said unanimously , these kids are indifferent to tests, as they do not care any more about success . They lost every bit of inner motivation to succeed in mathematics, and hence they could not care less about tests.

Conclusions

Even though we did not find a high level of test anxiety as we had theoretically predicted, we cannot recommend using achievement tests as the basis of curriculum evaluation for this particular population. As their teachers suggested, a major problem is that of total lack of success-drive. A careful study of this population's inner and external motives to succeed is needed. Meanwhile, innovative measures ought to be found in order to evaluate new mathematics curriculum, developed for these students. Perhaps, a measure of change in test anxiety can serve as an indicator of change in student's motivation, in case an increase in test anxiety is found.

References

Helmke, A. (1988) The Role of Classroom Context Factors for the Achievement - Impairing Effect of Test Anxiety. Anxiety Research - An International Journal. 1 (1), 37-52.

Hembree, R. (1988) Correlates, Causes, Effects and Treatment of Test Anxiety, Review of Educational Research, 58 (1), 47-77.

Michaelis, E. et al (1988) A Modular Kit for Diagnosis and Eva-luation of Test Anxiety, Psychology Research Workshop, Tel Aviv University. (in Hebrew)

Movshovitz-Hadar, N. et al. (1987) Low Achievers in Mathematics - A Prototype. Unpublished Report. Technion, Department of Education, Haifa. (in Hebrew)

Sarason, I.G. (1984) Stress, Anxiety and Cognitive Interference Reactions to Tests, Journal of Personality and Social Psychology, 46 (4), 929-938.

Sarason, I.G. (1988) Anxiety, Self-Preoccupation and Attention, Anxiety Research - An International Journal, 1 (1), 3-7.

Proofs that Prove and Proofs that Explain

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Abstract

A distinction is made between mathematical proofs that prove and mathematical proofs that explain: those that explain show not only that a statement is true, but also why it is true. It is then argued that proofs that explain should be favoured in mathematics education over those that merely prove.

In recent years many mathematics educators have actively reassessed the role of mathematical proof in various parts of the curriculum, and as a result there has been a trend away from what has often been seen as an over-reliance on formal proofs. In a desire to take into account the role of proof as a means of communication, and in recognition of the social processes that play a crucial part in the acceptance by mathematicians of a new result, educators have come to place greater emphasis on the concept of proof as "convincing argument."

The trend away from formal proofs in the curriculum, and the resulting search for alternative ways of demonstrating the validity of mathematical results in the classroom, have motivated a number of studies dealing with the problem of teaching proof. Leron (1983), concerned that most of the formal proofs found in textbooks do a poor job of communicating mathematical ideas, suggested that such mathematical presentations would be much more comprehensible if the proof were structured into short autonomous modules, each emphasizing one particular idea.

Deploring the teaching of geometry as narrow and overly concerned with deductive proof, Volmink (1988) believes that mathematics education would be better served if the curriculum were to place greater emphasis on the social criteria for the acceptance of a mathematical truth, at the expense of the purely formal ones. Movshovitz-Hadar (1988) elaborates upon six different ways of presenting theorems and six ways of presenting proofs, in an effort to enhance mathematical understanding through what she calls the "stimulating responsive method."

Alibert (1988), on the other hand, relies on the method of scientific debate, which provides students the opportunity to discuss the arguments made by a proof. In an extensive study of the processes involved in teaching a mathematical proof, Balacheff (1988) also points to the importance of creating classroom situations in which the student becomes aware of the complexity of the problem and of the necessity to produce valid arguments.

These ideas, and others not cited here, have made a substantial contribution to our understanding of the didactics of proof, and have permitted their authors to offer specific and interesting new ways of teaching proofs. In these discussions, however, a proof is viewed primarily as a valid argument, as opposed to an argument that must be both valid and explanatory. I believe it would be useful to introduce to the discussion an explicit distinction between proofs that prove and proofs that explain.

In this paper I first address this additional aspect of proof, namely proof as explanation, and then consider the implications of this view of proof for the handling of proof in the curriculum, suggesting that we should, whenever possible, seek to present to students the proofs that explain rather than those that only prove.

Explanation versus Validation

Both proofs that explain and proofs that prove are legitimate proofs. By this I mean that both fulfill all the requirements of a mathematical proof. Each serves to establish the validity of a mathematical assertion. Each consists of statements that are either axioms themselves or follow from previous statements, and thus eventually from axioms, as a result of the correct application of rules of inference. Each is recognized by the mathematical community as a valid proof (though there may be differences of opinion on the degree of rigour).

There is nevertheless a very important difference between these two kinds of proof. A proof that proves shows only that a theorem is true; a proof that explains also shows why it is true. A proof that proves may rely on mathematical induction or even on syntactic considerations alone. A proof that explains must provide a rationale based upon the mathematical ideas involved: the mathematical properties that cause the asserted theorem or other mathematical statement to be true.

The sense in which I use the term *explanation* is perhaps best clarified in contradistinction to that of Balacheff. In his analysis of the cognitive and social aspects of proof, Balacheff (1988) proposed the following distinctions:

> - We call an *explanation* the discourse of an individual who aims to establish for somebody else the validity of a statement. The validity of an explanation is initially related to the speaker who articulates it.
> - We call *proof* an explanation which is accepted by a community at a given time.
> - We call *mathematical proof* a proof accepted by mathematicians. As a discourse, mathematical proofs have now-a-days a specific structure and follow well defined rules that have been formalized by logicians (p. 2).

For Balacheff, then, a proof is an explanation by virtue of it being a proof. Yet surely a proof need have no explanatory power. One can even establish the validity of many mathematical assertions by purely syntactic means. With such a syntactic proof one can demonstrate that a statement is true without ever showing what mathematical property makes it true. Thus I prefer to use the term *explain* only when the proof reveals and makes use of the mathematical ideas which motivate it. Following Steiner (1978), I will say that a proof explains when it shows what "characteristic property" entails the theorem it purports to prove. As Steiner put it:

> ... an explanatory proof makes reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the results depends on the property. It must be evident, that is, that if we substitute in the proof a different object of the same domain, the theorem collapses; more, we should be able to see as we vary the object how the theorem changes in response (p. 143).

The following example will illustrate the difference between a proof that proves and a proof that explains:

Prove that the sum of the first n positive integers, $S(n)$, is equal to $n(n+1)/2$.

A proof that proves

Proof by mathematical induction:

> For $n=1$ the theorem is true.
>
> Assume it is true for an arbitrary k.
>
> Then consider:
>
> $S(k+1) = S(k) + (k+1) = n(n+1)/2 + (n+1) = (n+1)(n+2)/2$
>
> Therefore the statement is true for $k+1$ if it is true for k
>
> By the induction theorem, the statement is true for all n.

Now, this is certainly an acceptable proof: it demonstrates that a mathematical statement is true. What it does not do, however, is show why the sum of the first n integers is $n(n+1)/2$ or what characteristic property of the sum of the first n integers might be responsible for the value $n(n+1)/2$. In general, proofs by mathematical induction are non-explanatory.

Gauss's proof of the same statement, however, is explanatory, because it uses the property of symmetry (of the different representations of the sum) to show why the statement is true. It makes reference to the property of symmetry and it is evident from the proof that the results depend on this property.

A proof that explains

Gauss's proof is as follows:

$$
\begin{array}{rl}
S = & 1 + 2 + \ldots + n \\
S = & n + (n-1) + \ldots + 1 \\
\hline
2S = & (n+1) + (n+1) + \ldots + (n+1) = n\ (n+1) \\
S = & n(n+1)/2 \quad \text{QED.}
\end{array}
$$

Another explanatory proof of this same statement is, of course, the geometric representation of the first *n* integers by an isosceles right triangle of dots; here the characteristic property is the geometrical pattern that compels the truth of the statement.

Both Gauss's proof and the geometric representation show that an explanatory approach to proof in the classroom need not always entail doing away with legitimate mathematical proofs and relying on intuition only. What is required is the replacement of one kind of proof, the non-explanatory kind, by another equally legitimate proof which has explanatory power, the power to bring out the mathematical message in the theorem. In their paper "*Wann ist ein Beweis ein Beweis?*" (When is a proof a proof?) Wittmann and Mueller (1988) refer to these kinds of proof, in fact, as "clear-content proofs" (*inhaltich-anschaulich*), and furnish an interesting example. The challenge is to identify suitable explanatory proofs as alternatives to the many non-explanatory ones now in use.

One might ask whether an abandonment of non-explanatory proofs would not make the curriculum less reflective of accepted mathematical practice. It is certainly true that, far from making the mathematical content clear, many mathematicians have thought it necessary in constructing a proof to avoid any reference at all to mathematical content, sometimes through reliance on purely syntactic methods. To ensure the correctness of their proofs, they have consciously emphasized the deductive mechanism at the expense of the mathematical ideas.

As I have argued elsewhere (Hanna, 1983), however, mathematicians, including those who have recourse to purely syntactic methods, are nevertheless really more interested in the message behind the proof than in its codification and syntax, and they see the mechanics of proof as a necessary but ultimately less significant aspect of mathematics. Furthermore, as I have also argued, the significance of what is proved is given more weight than the very correctness of the proof. Thus there is no infidelity to the practice of mathematics if in mathematics education we focus as much as possible on good mathematical explanations (even at the expense of rigour), and highlight for the students in our proof of a theorem the important mathematical ideas that lead to its truth.

Implications for Teaching

As mathematics educators it is our mission to make students understand mathematics. It is my contention that in support of this mission we should give a more prominent place in the mathematics curriculum to proofs that explain. Such a focus is particularly important in teaching, because, unlike mathematicians, students of mathematics have yet to learn the relative importance of different mathematical topics and may easily be misled by a classroom emphasis on the deductive mechanism.

The first step in promoting understanding through explanatory proofs is, of course, to recognize that understanding is much more than confirming that all the links in a chain of deduction are correct, that in fact the completeness of detail in a formal deduction may obscure rather than enlighten, and that understanding requires some appeal to previous mathematical experience. In discussing the relationship between understanding and proof it is useful to keep in mind that mathematical arguments may have various attributes (such as convincing, precise, formal, explanatory), and that these attributes are often quite distinct.

References

Alibert, D. Towards new customs in the classroom. *For the learning of mathematics, 8*(2), 31-35.

Balacheff, N. (1988). *A study of pupils' proving processes at the junior high school level.* Invited paper presented at the joint international conference 66th NCTM and UCSMP Project, Chicago.

Hanna, G. (1983). *Rigorous Proof in Mathematics Education*. OISE Press.

Leron, Uri. (1983). Structuring mathematical proofs. *American Mathematical Monthly, 90*(3), 174-185.

Movshovits-Hadar, N. (1988). Stimulating presentation of theorems followed by responsive proofs. *For the learning of mathematics, 8*(2), 12-19.

Steiner, M. (1978). Mathematical explanation. *Philosophical Studies, 34*, 135-151.

Volmink, John. (1988). The role of proof in students' understanding of geometry. Paper presented at the AERA annual meeting, New Orleans.

Wittmann, E.C. & Mueller, G. (1988) *Wann ist ein Beweis ein Beweis?*. Paper presented at the annual meeting of mathematics teachers, Germany.

Fischbein's Theory: A Further Consideration

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This paper presents a theoretical framework for and some preliminary results of a study which aims to further investigate Fischbein's theory on multiplication and division concepts, with sophomore and senior preservice teachers and elementary school inservice teachers. The study uses an instrument which controls several confounding variables to reexamine the impact of number type on problem difficulty and to further investigate the following aspects: (a) the domain of the number system from which the problem quantities are derived--fractions versus decimals; (b) the "absorption effect" notion; (c) the relative robustness of the intuitive rules associated with Fischbein's intuitive models; and (d) the solution processes used by subjects to solve multiplication and division problems. It was found that inservice teachers who were highly successful at multiplication and division problems employ proportional reasoning, others have difficulty translating a correct problem representation into a correct mathematical sentence, and the rest attend only to the surface structure of the problem .

In an attempt to understand preservice and inservice teachers' concepts of multiplication and division, we designed and implemented a study which controls a wide range of confounding variables; these are the variables of structure (e.g., simple proportion versus multiple proportion, Vergnaud, 1983), text (e.g., mapping rule versus multiplicative compare, Nesher, 1988), context, and syntax described in Harel, Post, and Behr (1988a). The study consists of three components. The first deals with the impact of the propositional structure on the problem situation of multiplication and division problems (see Harel, Behr, and Post, 1988b); the second component reexamines the impact of the number type (whole number, non whole number greater than one, and positive number smaller than one) on the problem solution and further investigates several aspects of Fischbein, Deri, Nello, and

Marino's (1985) theory. The third components aims to reveal teachers' processes of thinking on both pedagogical and mathematical aspects of multiplication and division concepts. The subjects in this study are sophomore preservice teachers (N=113), senior preservice teachers (N=63), and elementary school inservice teachers (N=139). The analysis of the results from this study is underway; in this paper we will present a theoretical framework of the second component of the study along with some preliminary results.

Fischbein's intuitive models. According to Fischbein et al. (1985), the model associated with multiplication problems is repeated addition. This model leads subjects to intuit the rule that a multiplier must be a whole number and reinforces the misconception that the product must be larger than the multiplicand, or multiplication makes bigger (Bell Fischbein, and Greer, 1984; Bell, Swan, and Taylor, 1981; Hart, 1984). For division, Fischbein et al. suggested two intuitive models: partitive division and quotitive division. Associated with the partitive division model are two intuitive rules: the divisor must be a whole number and the divisor must be smaller than the dividend. These rules result in another intuitive rule that the quotient must be smaller than the dividend, or division makes smaller. The only rule associated with the quotitive division model is that the divisor must be smaller than the dividend.

Domain of the number system. The studies that address the incongruity between these intuitive models and the formal operations of multiplication and division problems involve only decimal numbers; the question of whether a similar incongruity exists in multiplication and division problems with fractions has never been directly addressed. There is a reason to believe, however, that fractions and decimals do not have the same effect on the difficulty of multiplication and division problems. A rationale for this is that a fraction, more than a decimal, can be viewed as an operator. As a result, it might be easier to identify relationships among problem quantities in multiplication and division problems in which the multipliers and divisors are fraction than in those in which they are decimals.

The absorption effect. Fischbein et al. (1985) suggested the notion of the "absorption effect," to conjecture that "when the whole part of a decimal is clearly larger than the fractional part, the pupil may treat it more like a whole number (as though the whole part 'masks' or 'absorbs' the fractional part)." To support their conjecture, they compared performance on several problem types: one with the decimal multiplier 3.25, one with the decimal multiplier 1.25 and four with the decimal multipliers 0.75 or 0.65. They found that compared to decimals like 0.75, 0.65, or 1.25, a decimal like 3.25 has a slighter counterintuitive effect when playing the role of multiplier. This finding raises several questions: (a) What is the conceptual base for the argument that the whole part 3 in the decimal multiplier 3.25 better "masks" or "absorbs" the fractional part 0.25 than 1 does in the decimal multiplier 1.25?; (b) does the "absorption effect" apply to decimal multipliers between 2 and 3 (e.g., 2.25)?; (c) are "large" decimals (e.g., 42.35) better conceived as multipliers than small decimals (e.g., 3.25)? (d) does the relative size between the whole part and the decimal part of a decimal multiplier play a role in the "absorption effect?" (e) does the "absorption effect" apply to fractions as well?

Levels of robustness. A clear result from the study by Fischbein et al. and others (e.g., Graeber, Tirosh, and Glover, 1989; Mangan, 1986) is that a non-whole-number multiplier differentially affects the relative difficulty of a multiplication problem depending on whether it is greater or smaller than one. This suggests that violations of the rule, "multiplier must be a whole number," are of two types: one is when the multiplier is greater than one and the other is when the multiplier is smaller than one. Thus, with respect to violation of the intuitive rules associated with multiplication, there are three classes of multiplication problems:

M(0). Problems which conform to the the multiplication model,

M(1.1). Problems in which the multiplier is a non-whole-number greater than one, these violate exactly one rule: "multiplier must be a whole number,"

M(1.2, 2). Problems in which the multiplier is a number less than one; these violate exactly two rules: "multiplier must be a whole number" and "multiplication makes bigger."

Surprisingly, the impact of the "absorption effect" has not been investigated with respect to the intuitive partitive division rule, divisor must be a whole number, even though the same argument made with the multiplier can be made with the divisor. To address this effect we suggest a similar refinement to that with non-whole number multipliers in order to distinguish between a non-whole-number divisor greater than one and a positive divisor less than one. As in the multiplication case, this refinement results in the following classification of partitive division problems:

P(0). Problems which conform to the the partitive division model,

P(1.1). Problems in which the divisor is a fractional number greater than one; these violate exactly one rule: "divisor must be a whole number,"

P(2). Problems which violate exactly one rule: "divisor must be smaller than dividend,"

P(1.1, 2). The intersection of the classes P(1.1) and P(2); that is, problems which violate exactly the two rules: "divisor must be a whole number" and "divisor must be smaller than the dividend."

P(1.2, 3). Problems in which the divisor is a fractional number smaller than one; these violate exactly two rules: "divisor must be a whole number" and "division makes smaller,"

P(1.2, 2, 3). The intersection of the classes P(1.2, 3) and P(2); that is, problems which violate all three rules.

Since quotitive division problems are associated with only one rule, the divisor must be smaller than the dividend, their classification is:

Q(0). Problems which conform to the the quotitive division model.

Q(1). Problems which violate the rule: "divisor is greater than the dividend."

This analysis and a careful examination of results from different studies led us to hypothesize that the intuitive rules are not equally robust in problem solutions. For example, from the results reported by Fischbein et al. (1985) we hypothesized that different intuitive rules within the partitive model may not be equally strong in affecting students' solution of partitive division problems: children prefer to cope with the violation of the rule, "divisor must be smaller than the dividend," than with the violation of the rule divisor must be a whole number (see Harel et al. 1988b).

A first indication of how the intuitive rules associated with the partitive division model differentially affect the solution performance can be seen from Table 1. Performance on P(2) problems--those which only violate the rule, "dividend must be greater than divisor"--is higher than performance on the other categories of problems which violate one or a combination of intuitive rules. This indicates that the rule, "dividend must be greater than divisor," is the least robust in the solution of partitive division among other combinations of the intuitive rules. Further indication of the relative robustness of the intuitive rules associated with the partitive division model will discussed below.

Another important result from Table 1 is that the "absorption effect" does not apply to the divisor in partitive division problems. This can be seen by comparing the performance on P(1.1) problems (whose divisor is a decimal greater than one) to the performance on two classes of problems: P(0) (problems whose divisor is a whole number) and P(1.2, 3) (problems whose divisor is a decimal smaller than one). Table 1 shows a big difference in the first comparison, which suggests that, unlike a decimal multiplier greater than one, a decimal divisor greater than one is not treated like a whole number. The second comparison shows no precedence of a decimal divisor greater than one to a decimal divisor smaller than one, and since performance on both classes is low, both impose a strong constraint that causes major difficulty in problem solution.

Table 1

Category	Rule violation	Operation	Correct responses (%)		
			Preservice		Inservice
			Sophomore	Senior	
P(0)	No rule violation	68÷17	86.5	92.5	95.5
P(1.1)	Divisor must be a whole number	11÷2.53	31.5	36	41
P(2)	Dividend must be greater than divisor	3÷5	62.3	79.7	85
P(1.1, 2)	Divisor must be a whole number AND Dividend must be greater than divisor	12÷24.67	19	26.7	41
P(1.2, 3)	Divisor must be a whole number AND Quotient must be greater than dividend	6÷0.67	39.3	52	61
P(1.1, 2, 3)	Divisor must be a whole number AND Dividend must be greater than divisor AND Quotient must be greater than dividend	0.35÷0.79	33	43	55

A further distinction among the intuitive rules described above is that some of the rules are associated with the problem information, others with the problem solution. In multiplication, the rule, "multiplier must be a whole number" imposes a constraint on the type of multiplier provided in the problem information; in contrast, the rule, "multiplication makes bigger" restricts the problem solution to be a number greater than the multiplicand. Similarly, in partitive division, the rules, "divisor must be a whole number" and "divisor must be smaller than dividend" are problem information rules, whereas the rule "quotient must be greater than dividend" is a "problem solution" rule. Finally, the rule, "divisor must be smaller than dividend" associated with quotitive division is a "problem information" rule.

Any problem solution rule is dependent on a problem information rule; therefore, the relative robustness of these two types of rules cannot be tested independently. However, we hope to get some information about the role that these two types of rules play in the solution process from the interview data.

Summary

In this paper we have addressed several aspects of Fischbein et al.'s (1985) theory: (a) the number type and the domain of the number system from which the problem quantities are derived--fractions versus decimals; (b) the "absorption effect" notion suggested by Fischbein et al. to account for differences in subjects' performance on multiplication problems with multiplier greater than one versus those with multiplier smaller than one; and (c) the relative robustness of the intuitive rules associated with the intuitive models suggested by Fischbein et al.

The primary results described in this paper are very discouraging: preservice and inservice teachers have serious misconception in a content domain that is included in their teaching responsibility. Moreover, in comparing teachers' misconceptions to those possessed by children (Bell, Fischbein, and Greer, 1984; Bell, Swan, and Taylor, 1981; Fischbein et al. 1985; Hart, 1984; Mangan, 1986), and teachers' solution strategies to those used by children (e.g., Sowder, 1988) we found a striking resemblance. Graeber et al. (1989), who found this same result, indicated that " efficient strategies are needed for training teachers to monitor and control the impact that misconceptions and primitive models have on their thinking and their students' thinking" (p. 100). Graeber et al. suggest such strategies. For example, asking teachers to write about their conception and misconception, or encourage them to compare their estimated answer with the computed one. These strategies can be used with children as well. Fischbein et al. (1985) recommended that teachers (assuming they are competent in the content domain) should "provide learners with efficient mental strategies that would enable them to control the impact of the primitive models" (p. 16). Greer (1985) recommended that teachers should "aim to widen the range of models available to the pupils" (p. 74).

Our investigation of the solution processes used by inservice teachers suggests that the main difficulty is in the translation process of the comprehension representation (i.e., the understanding the relationships between the problem quantities, see Harel and Behr, in press) into a mathematical representation (i.e., a mathematical expression that represent these relationships). High performance teachers use the concept of proportion in employing this translation process by representing multiplication and division problems as missing value proportion problems. Intuitive solution strategies are available to this representation. These strategies involve determining the multiplicative relationship between two given quantities and extending that relationship to the other two quantities to find the unknown quantity (see Harel and Behr, in press; Vergnaud, 1983). Sellke, Behr, and Voelker (1988) show that this approach results in improved performance with seventh-grade children.

References

Bell, A., Fischbein, E., and Greer, B. (1984). Choice of operation in verbal arithmetic problems: the effect of number size, problem structure and context. Educational Studies in Mathematics, 15, 129-147.

Bell, A., Swan, M., and Taylor, G. (1981). Choice of operations in verbal problems with decimal numbers. Educational Studies in Mathematics, 12, 399-420.

Fischbein, E. Deri, M., Nello, M. & Marino, M. (1985). The rule of implicit models in solving verbal problems in multiplication and division. Journal for Research in Mathematics Education. 16, 3-17.

Greer, B. (1985). Understanding of arithmetical operations as models of situations. In J. Sloboda and D. Rog (Eds.) Cognitive Processes in Mathematics. London, Oxford University Press.

Harel, G., & Behr, M. (in press). Structure and hierarchy of missing value proportion problems and their representations. Journal of Mathematical Behavior.

Harel, G., Behr, M. & Post, T. (1988a). On the textual and semantic structures of Mapping Rule and Multiplicative Compare Problems. The Proceedings of the 12-th Annual Conference of the PME, Hungar

Harel, G., Post, T., Behr, M. (1988b). An assessment instrument to examine knowledge of multiplication and division concepts and its implementation with inservice teachers. The Proceedings of the 9-th Annual Conference of the PME-NA, DeKalb, Illinois.

Hart, K. M. (1984). Ratio: Children's strategies and errors in secondary mathematics project. Windsor, Berkshire: NFER-NELSON.

Mangan, C. (1986). Choice of operation in multiplication and division word problems. Unpublished doctoral dissertation, Queen's University, Belfast.

Nesher, P. (1988). Multiplicative school word problems: Theoretical approaches and empirical findings. In J. Hiebert and M. Behr (Eds.) Number Concepts and Operations in the Middle Grades. Reston, VA, Nation Council of Teachers of Mathematics.

Graeber, A., Tirosh, D., & Glover, R. (1989). Preservice teachers' misconception in solving verbal problems i multiplication and division. Journal for Research in Mathematics Education. 20, 95-102.

Sowder, L. (1988). Children's solutions of story problems. Journal of Mathematical Behavior, 7, 3.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh and M. Landau (Eds.) Acquisition of Mathematics Concepts and Processes. NY, Academic Press.

THEY'RE USEFUL - CHILDREN'S VIEW OF CONCRETE MATERIALS

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The provision of a bridge between the use of concrete materials and formal mathematics was tested with 12 teachers. Their interpretation of this distinct activity is reported.

When children are taught mathematical rules by appealing to the results of concrete experiences, their teachers often tell them that the materials are helpful for understanding. The pupils later repeat this claim of helpfulness although they themselves are unable to display success with the materials. Diagrams appear more useful than rods in the solution of equations but not very useful when equivalent fractions are taught.

This report extends the statement made at PME, 1988 on the research, funded by Nuffield and carried out at King's College, London, on the connections between the use of concrete materials and formal mathematics. The particular aspect of using concrete materials, considered in the research project "Children's Mathematical Frameworks" (Hart et al, in press) was one which is very commonly used in upper primary and early secondary school classes. The pupils are given structured situations in which they use manipulatives, from the results they are asked to see patterns which are then formalised in rules, algorithms or formulae.

The research reported here concerned the same type of situation. The emphasis was however, on the effectiveness of the imposition of a third type of activity which was neither using concrete material nor formal mathematics but formed a 'bridge' between them. The volunteer teachers were asked to teach two matched classes. With the second of these the normal progression to the formalisation would include a 'bridge'. The teachers, topics and time spent on them are shown in Table 1.

Table 1
Topics, teaching time and 'bridge' activities

Teacher	Topic	Time spent on teaching scheme	Bridge Activity
A	Equations	10 hours	Diagrams
B	Equations	6 hours 40 minutes	Child discussion
C	Equations	7 hours	Child discussion
D	Volume	2 hours 55 minutes	Child discussion
E	Volume	2 hours 55 minutes	Table
F	Volume	2 hours 55 minutes	Table
G	Volume	2 hours 55 minutes	Graphs
H	Volume	4 hours	Table
I[a]	Equivalent Fractions	?	Child discussion
J	Equivalent Fractions	6 hours	Table
K	Equivalent Fractions	4 hours	Table
L	Area of a rectangle	5 hours	Table

[a] Data provided by this teacher were incomplete and insufficient for analysis

The Bridge

In the research, the 'bridge' was described as an activity which was distinctly different from both the concrete materials and the formalisation but was seen to connect the two. The plans of the 12 teachers were examined and the following were thought to embody the 'bridge' criteria.

a) Child discussion - the emphasis being on the pupils expressing themselves and not simply answering questions posed by the teacher. Such a discussion could also include ideas on examples counter to the formalisation.

b) Tables - the concrete experiences often gave rise to results which could be put in a table, the pattern emphasised and the generalisation ensue.

c) Graphs - similar to tabulation was graphing, in which results were recorded on squared paper, a pattern sought and a relationship expressed.

d) Diagrams - this was possibly the activity closest to the use of concrete materials since it was essentially to represent the manipulatives and what was done with them, through diagrams.

After the first class had been taught the researchers discussed which 'bridge' could be used for the second class, with the teacher, who then planned the activity, as shown in Table 1.

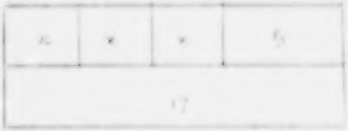
The Teaching

The effectiveness of two teaching sequences was under investigation. They were designed to be identical as far as the concrete material and formalisation phases were concerned. Only one lesson in each sequence was observed by a researcher who took notes of blackboard displays etc., and tape-recorded the statements of the teacher. No teacher explained to the class that a method of solution which was generalisable to many situations was a very powerful mathematical tool although two said it would be quicker to use than the concrete material.

Having chosen which style of 'bridge' was to be used, the teacher designed the content of it.

'Child Discussion was used by two of the teachers who were concerned with the solution of algebraic equations. In both cases, however, it came after the formalisation so was a discussion of the method rather than a verbalisation of the connection between materials and method.

Diagrams of algebraic equations were used as a 'bridge' by one teacher and in this case they proved to be a greater prop to the children than the actual materials. There were two reasons for this, one being the fact that it is easier to draw a diagram freehand (and inaccurately) than to find bricks to line up. Secondly and possibly more important is the fact that diagrams can represent lengths of any size and are not restricted to the preordained lengths of Cuisenaire rods or Colour Factor. To show for example $3x + 5 = 17$, the fact that there is no rod of length 17cm, is immaterial, one draws a long rectangle and simply labels it 17, as shown here.



Tabulation seemed to be a natural way of recording results from concrete experiences in finding a) the volume of a cuboid or b) equivalent fractions. In each case the patterns of numbers were meant to suggest a rule. Three teachers chose this bridge for 'Volume', two chose it for 'Equivalent Fractions' and one used it in the teaching of the Area of a Rectangle. In no class, however, were the tables emphasised, discussed at length, put forward as a good way of presenting information or indeed explained as a way of connecting the blocks (in Volume) and the formula. In three cases the tables were presented on worksheets and not subsequently mentioned in the lesson.

In only one case were the results graphed. This was by a teacher of the Volume formula. There is a drawback to its use here - one is only able to draw a graph of the relationship between two variables and in $V = L \times B \times H$ we have four, so we are restricted to a fixed situation (such as a layer of constant area) rather than illustrating the general case.

The time spent on the setting-up, exploitation of the bridge and in linking it to both concrete situation and formalisation was in the case of 'tables' and 'graphs' very short. Six teachers spent ten minutes or less on this linking activity, see Table 2.

Table 2

Time spent on 'connecting' activity in formalisation and 'bridge' lessons

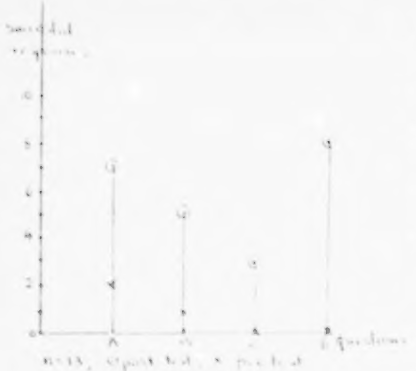
Teacher	Formalisation lesson	'Bridge' lesson	
A	60	60	minutes
B	35	35	
C	20	60	
D	40	30	
E	8	35	
F	5	1	
G	10	5	
H	40	2	
J	30	10	
K	6	4	
L	10	8	

Child Interviews

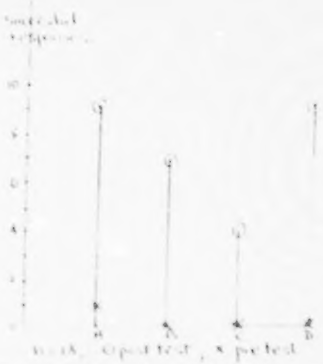
The teachers were asked to interview six children from each group, just after the teaching, in order to obtain further information on their understanding and to illuminate the post-test scores. The researchers interviewed the same children three months after the teaching. The questions were intended to provide information on (i) the methods used by the children to solve problems (the formalisation or something else), (ii) their attitude to and use of the concrete materials (iii) their memory of how the formalisation was arrived at and (iv) their appreciation of the connection between the two (or three) phases of teaching.

The immediate post tests showed very little difference in performance between the two groups in each topic see Fig 1 for some typical examples.

Fig 1



Class taught without 'bridge' activity: Teacher J.



Class taught with 'bridge' activity: Teacher J.

Similarly the interviews produced no vastly different response from the 'bridge' children than from the others. Over half the number of children interviewed stated that the materials were helpful. Asked about the rods used in the solution of equations children responded:-

When using the rods you can actually see what you're doing and actually take them away and move them. (Bethan)

Well using the rods is easier if you're got big numbers of x's or something. (Bethan again)

It's a lot easier to begin with, it's a lot easier with bricks (Ross)

It's easier to do with blocks (John)

Oh bricks, it's easier to understand. I suppose that those make it easy to explain what you're doing here. (Helen)

Well it's just that's on paper and that's kind of real, you can see it, you can move it about and it helps you more. (Giles)

To explain to people I'd use bricks because you can actually see what is happening, you can see what you're taking away. (David)

These same children, however never chose to demonstrate the solution of equations with rods and were unable to do so when asked. Three (out of 12) could set up the equation.

One teacher of equations had introduced the use of diagrams alongside the bricks to both her classes whilst another teacher had diagrams as the 'bridge'. Fifteen (out of 18) of these users thought diagrams were helpful and eight of these could set up the equation with diagrams and make an attempt at solution. Diagrams seem to provide a better support than the bricks themselves.

The embodiment used by the two teachers dealing with equivalent fractions was the diagram of a region. On the three month interview 17 pupils were asked to show $3/4 = 6/8$ using diagrams. Seven of them could do so, although all the diagrams were inaccurate. Only two chose to use diagrams to demonstrate the equivalence. Indeed, only half of those asked, said that a diagram would be helpful.

Further Research

There appears to be evidence that children do not use the concrete embodiment, on which the teaching was based, after the formalisation has been taught. This does not mean that the formula or algorithm is necessarily available to them, often neither experience has provided a usable skill. Those interviewed however, avowed that the concrete materials were useful.

CHILDREN'S INDIVIDUALITY IN SOLVING FRACTION PROBLEMS

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In interviews, 24 pupils (aged 11 to 13) solved fraction problems; in addition, the technique of "concept mapping" was used to find out which individual fraction concepts had been constructed by the children. Some examples of pupils' solutions and concept maps are given.

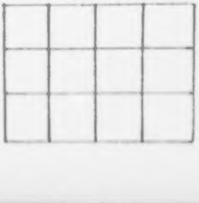
To interpret the results, the problem solving processes are regarded from three points of view: pupil's problem representation, cognitive style, and "conceptual world". Each aspect is indicated by contrasting prototypes which are used to characterize different kinds of behaviour, and to explain the great variety in individuals' problem solving behaviour.

Fractions are regarded as a rather difficult subject in mathematics teaching. In our country, pupils do not have formal instuctions on fractions until they are in grade 5 or 6, then from the beginning on a rather abstract level. It is, of course, intended that the children get "relational understanding" of fractions (cf. Skemp, 1979). A problem is how to check whether they have reached this goal or not.

In interviews, we gave (word) problems to the pupils and, in addition, we used the technique of "concept mapping" (see below). This paper is to show how differently - and how idiosyncraticly - the pupils proceeded when they solved fraction problems, and to explain why there is such a big variety in the pupils' problem solving behaviour.

After they had had formal instructions on fractions, 24 pupils (of grade 6, aged 11 to 13) were interviewed. They were asked to solve three problems on the addition of fractions (the pupils was given the choice of item 1 or 1*, resp.):

1. At first, shade in $\frac{3}{4}$ of the rectangle then shade in $\frac{1}{6}$ of the rectangle as well.
   What fraction of the whole rectangle did you shade in altogether?



1.* At first, mark $\frac{3}{4}$ of the straight line. Then mark $\frac{1}{6}$ of the straight line as well. When you put these two parts of the line together: what fraction of the whole line form these two parts altogether?

2. Calculate: $\frac{3}{4} + \frac{1}{6}$

3. Mother wants to divide out 4 apples to her 4 children. Unfortunately, one apple has a bad patch.
So mother gives at first 3 apples to the 4 children; after that each child gets one sixth of the fourth apple.
What fraction of an apple does each child get altogether?

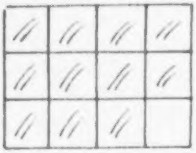
Obviously, these items are equal in content: one has to add $\frac{3}{4}$ and $\frac{1}{6}$ (some children, however, did not realize this fact). From an adults point of view, item 1 seems to be a very easy one as the rectangle is already divided in 12 equal parts (the straight line has a length of 12 cm). The calculation in item 2, in fact, turned out to be rather easy for this sample of pupils, whereas the problem in item 3 consciously was worded a little complicatedly as it is well-known that in word problems a pupil's selection of arithmetical operations mainly depends from the kind in which he or she constructs a mental image of the situation described in the problem (see, e.g., Fischbein et al., 1985, or Greer, 1987).

The technique of concept mapping (see Novak et al., 1983) was used to find out which individual fraction concepts had been constructed by the pupils; it was used, however, different from the way it was described by Novak et al.: Twelve small cards with concept names like "fraction", "rational number", "numerator", "denominator", "$\frac{1}{6}$", "$\frac{3}{4}$", "one sixth", "apple", "rectangle" (or "straight line"), "to divide out", etc. (see fig. 2 and 4) were given to the children, and they were asked to distribute these cards on a sheet of paper in such a way that concepts which are related were put together closely whereas those concepts which are not related were separated on the sheet. In a second step, the children were asked to find a generic term for the chunks of concepts and/or to mark and to name relationships between the concepts on the sheet.

Examples for pupils' solutions (the children's solutions are indicated by hand-writing):

Till:

ad 1:



$\frac{11}{12}$

fig. 1

ad 2: $\frac{3}{4} + \frac{1}{6} = \frac{9}{12} + \frac{2}{12} = \frac{11}{12}$

ad 3:

~~$4 : \frac{1}{6} = \frac{4 \times 6}{1 \times 1} = \frac{24}{1}$~~

$\frac{4}{1} : \frac{3}{1} = \frac{4 \times 1}{1 \times 3} = \frac{4}{3}$

$\frac{4}{3} + \frac{1}{6} = \frac{8}{6} + \frac{1}{6} = \frac{9}{6} = \frac{3}{2} = 1\frac{1}{2}$

From Angela's solution just her concept map will be presented; this map obviously is very much influenced by the problems she had solved before:

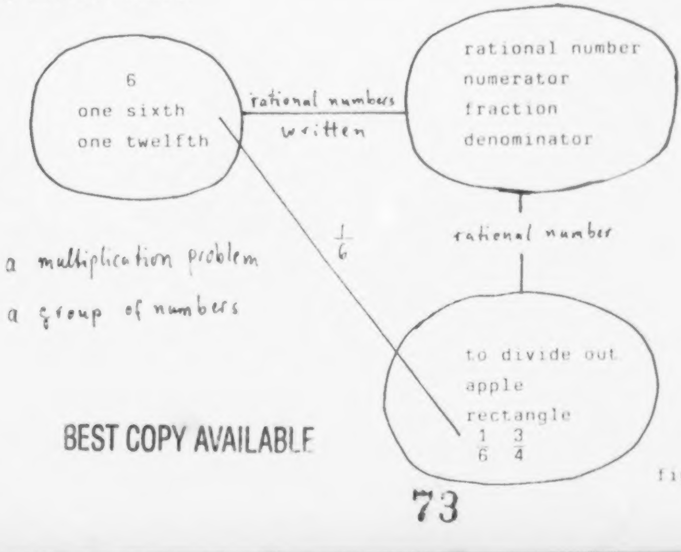

6
one sixth
one twelfth
rational numbers
written
rational number
numerator
fraction
denominator
$\frac{1}{6}$
rational number
a multiplication problem
a group of numbers
to divide out
apple
rectangle
$\frac{1}{6}$ $\frac{3}{4}$

fig. 2

Andreas took the straight-line-version of item 1; he marked 3 cm for $\frac{3}{4}$ and 2 cm for $\frac{1}{6}$ of the line. His answer "$2\frac{1}{6}$" refers to the whole line: The whole line is twice as much as the new line (2 x 5 cm) plus a remainder of 2 cm ($= \frac{1}{6}$).

After he had done item 2, Andreas realized his mistake:

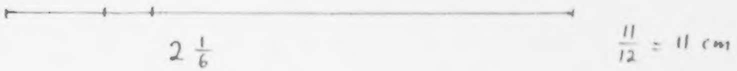

$2\frac{1}{6}$
$\frac{11}{12} = 11$ cm

fig. 3

ad 2: $\frac{3}{4} + \frac{1}{6} = \frac{18}{24} + \frac{4}{24} = \frac{22}{24} = \frac{11}{12}$

ad 3: Andreas emphasized that he had constructed a mental image of the apples. In this way, he had "seen" that this problem was already solved by the calculation in item 2.

In the concept mapping experiment, however, Andreas did not at all refer to the problems: He put together all concepts related to the fraction concept, but he separated the other concepts and claimed that, for instance, "to divide out apples belongs to (the subject) German".

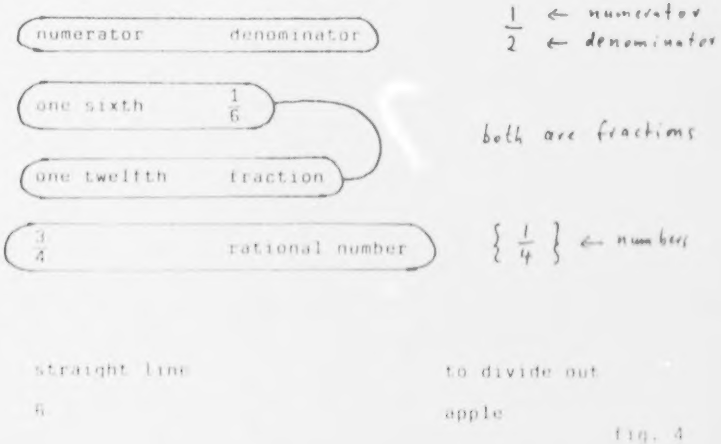

numerator
denominator
$\frac{1}{2}$ ← numerator ← denominator
one sixth
$\frac{1}{6}$
one twelfth
fraction
both are fractions
$\frac{3}{4}$
rational number
$\{\frac{1}{4}\}$ ← numbers
straight line
6
to divide out
apple

fig. 4

From Nils and Nike just the answers to item 1 will be given:

Nils:

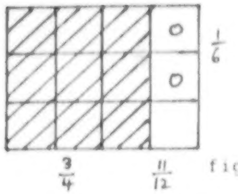


fig. 5

Nike:

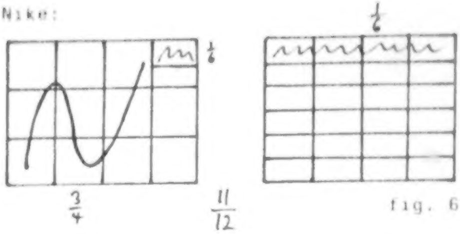


fig. 6

For an analysis of pupils' solutions it would be appropriate to describe each pupil's individual "fraction frame" (cf. Davis, 1984, or Hasemann, 1986a,b) in more detail. Andreas, for instance, when solving the word problem in item 3 linked together the "subunit-frame" (see Hasemann, 1986a, p. 135, or 1986b, p. 63) with a correct interpretation of "$\frac{1}{6}$ of the line" in a rather strange way.

However interesting pupils' individual frames are, in the following to explain children's individuality some **prototypes** of problem solvers will be discussed. To give characteristics of these prototypes, we shall look at the problem solving processes mentioned above from three different points of view:

(i) The kind of problem representation in a pupil's mind,
(ii) a pupil's cognitive style, and
(iii) the kind of a pupil's "conceptual world"

(whereas (i) and (ii) refer to an actual problem solving prosess, (iii) names a more general disposition; ad (i) and (ii) see also Cohors-Fresenborg & Kaune, 1984, and Schwank, 1986).

ad (i): When actually solving a problem there are two types of pupils: Type A **sees** the problem and its solution; type B **calculates** the result.

"To see" means here: to construct a mental image of the situation described in the problem. In item 1, for instance, this can be done in a rather natural way as the fraction that is asked for can be identified just by looking at the the rectangle (Nike's solution is a very nice example for this kind of behaviour, see fig. 6). On the other hand, there were pupils who calculated; Heike, in the staight-line-version (item 1*), for instance, did like this:

$$\frac{3}{4} \text{ of } 12 \text{ cm} = \frac{3}{4} \times 12 = \frac{3 \times 12}{4 \times 1} = \frac{9}{1} = 9 \text{ cm}$$

$$\frac{3}{4} + \frac{1}{6} = \frac{9}{12} + \frac{2}{12} = \frac{11}{12} \text{ cm}$$

Pupils of type A or B, resp., can also be identified with the word problem in item 3: Andreas (see above) is an example for type A. Type B - pupils, on the other hand, try to find out what they can **do** with the figures which are given in the problem (as an example see Till).

Regarding these examples, the characteristics of types A and B can also be described als follows: Type A - pupils are mainly interested in the problem itself, i.e. in the situation given in the problem and/or in its conceptual structure. Type B - pupils, in comparison, are mainly interested in acting: Which arithmetical operations or procedures match the situation (or even just the figures) given in the problem?

ad (ii): When observing the pupils' problem solving processes, Kogan's differentiation of **reflexive** and **impulsive** children (see Kogan et al., 1966) obviously makes a lot of sense (for example, look at Nils' and Nike's solutions in item 1).

ad (iii): A Pupil's problem solving behaviour seems to be highly influenced by the way he or she **thinks about** mathematical concepts: Some children seem to ignore the situation given in a problem, but right from the beginning they relate the problem to the conceptual framework they have (already) in mind (we call them **concept-orientated** pupils). Some others prefere to think about situations; they relate concepts to situations they have in mind (we call them **context-** or **task-orientated**). The former pupils are easily to recognize by their concept maps: All concepts related to the concept of fractions are grouped together, but concepts like "rectangle" or "apple" are excluded (see Andreas in fig. 4). The latter pupils try to construct mental images of the situations described in the problems, and their concept maps are representations of these mental images (see Angela in fig. 2).

(It should be remarked that the contrasting of concept- vs. context-orientated children does **not** refer to the fact that an individual's interpretation of a mathematical concept always depends from his or her "domain of subjective experience" (see Bauersfeld, 1983, Kilpatrick, 1985, p. 19f, or Hasemann, 1988, p. 128f). It should also be remarked that child's view of the (mathematical) world seems not to depend from age or mathematical ability.)

Obviously, the prototypes and categories mentioned above do not match each child's individual problem solving process. It was, however, possible to illustrate the contrasting types by some characteristics and examples. If it is accepted that the contrasts which were given make sense, and that the aspects (i) to (iii) are - more or less - independend, then there are at least 2 x 2 x 2 = 8 different kinds of problem solving behaviour that explain the big variety in this behaviour. But much more important is the fact that each child in his or her problem solving behaviour is an individual who has a right to be accepted as such, and to be treated adequatedly.

Regarding aspect (iii), for example, in our experiment most pupils turned out to be concept-orientated. When these children have difficulties with a word problem it makes not much sense to get them to look at the problem again and again, and to ask them to construct a mental image of the situation which is described in the problem - that's not the world they feel at home in. Instead, the teacher should try to enable these children to relate the problem to the conceptual framework they have already constructed, or to check whether their results are reasonable or not, i.e whether the results can be accepted considering the conceptual framework which was used to solve the problem. If, for instance, a fraction is asked for and an integer figures out one should become suspicious.

Anyhow, we as researchers and teachers have to accept that there are individuals who are concept-orientated and who try to ignore the context of a task. We should try to find out whether there are aspects in out teaching that have caused this fact or whether these preference is independent from the kind of teaching. The technique of concept_mapping seems to be rather effective to elucidate a child's conceptual world; in addition, concept maps are a very useful tool to become aware of children's alternative conceptual framework.

References

Bauersfeld, H. (1983): Subjektive Erfahrungsbereiche als Grundlage einer Interaktionstheorie des Mathematiklernens und -lehrens. Bauersfeld, H. et al.: *Lernen und Lehren von Mathematik*. Köln: Aulis (IDM-Reihe, vol 6), p 1-56.

Cohors-Fresenborg, E. & Kaune, C. (1984): Sequential versus conceptual. Two modes in algorithmic thinking. *Proc PME 8*, (Sydney, 16.-19.8.1984), p 261-267.

Davis, R.B. (1984): *Learning mathematics*. Beckenham, Kent: Croom Helm.

Fischbein, E. et al. (1985): The role of implicit models in solving decimal problems in multiplication and division. *J. for Research in Mathematics Education*, vol 16, p 3-17.

Greer, B. (1987): Non conservation of multiplication and division involving decimals. *J. for Research in Mathematics Education*, vol 18, p 37-45.

Hasemann, K. (1986a): *Mathematische Lernprozesse - Analysen mit kognitionstheorerischen Modellen*. Braunschweig: Vieweg.

Hasemann, K. (1986b): Analysis of fraction errors by a model of Cognitive Science. *European Journal of Psycholgy of Education*, vol 1, no 2, p 57-66.

Hasemann, K. (1988): Kognitionstheoretische Modelle und mathematische Lernprozesse. *Journal für Mathematik-Didaktik*, vol 9, no 2/3, p 95-161.

Kilpatrick, J. (1985): Reflection and Recursion. *Educational Studies in Mathematics*, vol 16, p 1-27.

Kogan et al. (1966): Conceptual impulsivity and inductive reasoning. *Child Development*, vol 37, p 583-594.

Novak, J.D. u.a. (1983): The use of concept mapping and knowledge Vee mapping with Junior High School science students. *Science Education*, vol 67, p 625-645.

Schwank, I. (1996): Cognitive structures of algorithmic thinking. *Proc PME 10* (London, 20.-25.7.86), p 195-200.

Skemp, R.R. (1979): Goals of learning and quality of understanding. *Math. Teaching*, no 88, p 44-48.

ASPECTS OF DECLARATIVE KNOWLEDGE ON CONTROL STRUCTURES

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Knowledge has procedural aspects as well as declarative aspects, which means, that there is a knowledge of concepts as well as a knowledge of rules applied while working with a specific concept. In particular, computer programming presupposes a knowledge of declarative and procedural aspects of control structures. In order to assess declarative knowledge on control structures, we used the method of concept mapping (Novak, Gowin, & Johansen, 1983) with students of lower secondary level after some months of programming instruction. It was applied to concepts representing knowledge on iteration and recursion.

Procedural and declarative knowledge

Knowledge diagnosis and knowledge representation have become central problems of cognitive science researchers and computer scientists as well. There is an intensive discussion of these topics in both disciplines. For computer scientists, the main task is representing expert knowledge in a machine adapted way. In particular, knowledge representation is a problem of defining and accessing data structures (Shapiro, 1987). A symbolic representation system has to be designed which fits into the specific pieces of knowledge and allows a mapping of its structure in a machine. However, it is not only the technological prerequisites of this research which have to be taken into account. Obviously, in behalf of computer developments, there has been tremendous effort and success in recent years. In contrast to these encouraging results concerning the hardware equipment, work on the problem of diagnosing knowledge (Tergan, 1988) is still in its beginnings. Research in cognitive psychology is mainly concerned with modelling cognitive processes, and, in particular, with modelling components of memory (Tack, 1987). Applications for these research results may be found in the development of intelligent tutoring systems. Such systems are aiming at an acquisition of expert knowledge in a specific domain with the help of an interactive computer program. They take into consideration the student's success while working with the program and so show need for a separate component of knowledge diagnosis. Having diagnosed the knowledge, the difference between the goal of a learning process and the actual state of the learner causes execution of a specific part of the learning program (Lesgold, 1988).

With respect to a discussion in computer sciences (Winograd, 1975) and to Anderson (1983) and his theory of thought, procedural and declarative aspects of knowledge may be differentiated. Crucial concepts in this respect are the working memory and the long term memory. The working memory consists of elements which are directly accessible at a certain instant. These elements are knowledge structures of temporary importance as well as activated particles of the long term memory. Moreover, there is a possibility for conclusions provided in the working memory. In particular, Anderson presumes that working memory and short term memory are not identical, which is contrary to older theories of thought (cf. Miller, 1956). The working memory interacts with both the short term memory and long term memory. For our purposes, the interaction with the long term memory is of special importance. This part of the memory includes procedural as well as declarative elements. Procedural elements or productions indicate which action has to ensue from a specific condition. Thus, productions are sometimes referred to as condition action pairs. *Applying knowledge* in this framwork usually means sequencing a number of productions, and so *representing knowledge* may be performed by constructing a production system which consists of all the independent rules leading to the solution of a problem. In this process, not only procedural aspects but also declarative components of knowledge are involved. Establishing a condition for a possible action includes asking for certain properties of a piece of information. These properties are part of the declarative knowledge which is also situated in long term memory. They are organised in such a way that classes and subclasses may be distinguished. So every concept is connected with certain specific properties, but it is also connected with properties which are true for a whole class of concepts. This way of storage has economical advantages, because every property is stored only once in connection with the most general subclass a concept belongs to (Anderson, 1983).

So, declarative knowledge includes not only specific concepts but also connections between those concepts. According to Anderson (1983) declarative knowledge may be represented as sequences in time, images in space, abstract propositions, or a combination of these elements. In the following, we are primarily interested in abstract propositions. Their objects are concepts and their relations are semantic relations between the concepts. Giving a representation of an abstract proposition might lead to a semantic network. A semantic network is a graph consisting of nodes which are connected by directed edges. The nodes stand for concepts, the edges represent relations between these concepts, the direction gives the distinction between the subject and the object of the proposition (Wender, 1988).

Assessing declarative knowledge

Different methods of representing declarative knowledge in a semantic network which will be described shortly in the following text have been established. We will present the methods of Novak, Gowin, & Johansen (1983), Ballstaedt & Mandl (1985), Scheele & Groeben (1984), and Feldmann (1979).

Novak, Gowin, & Johansen (1983) presented a method they refer to as *concept mapping*. They aimed at investigating declarative knowledge in physics, and presented a certain number of concepts concerning this subject written on small cards to their students. The students were supposed to group the cards so that similar concepts were close to each other. Moreover, the subjects were asked to give a verbal description of relations they realized between different concepts. The work results in a concept map representing declarative knowledge as a complex network of concepts. A very similar method was worked out by Ballstaedt & Mandl (1985). In contrast to Novak & al., Ballstaedt & Mandl do not use a fixed number of concepts. Their subjects are not only supposed to find relations between presented cards, but were asked to add new concepts to their map as well. The number of concepts involved in a map gives evidence for its quality, because experts in a specific domain use a wider range of concepts than novices. The design of Scheele & Groeben (1984) is also based on a varying number of concepts. But the student is supposed to use only certain relations and so match concepts with respect to these relations. Feldmann (1979) does not pay attention to the semantics of a relation but only to the number of connections between concepts. The only criterion for assessing knowledge is the existence and direction or the non-existence of an edge between two nodes.

Because we were interested in assessing knowledge in the domain of control structures with two students who had attended a Logo course for some months, we decided to use the concept mapping method of Novak & al. (1983), which had proved to be appropriate for mathematics related concepts (Reiss, 1987). We were primarily interested in the way students looked at concepts related to iteration or recursion. Our hypotheses was that their understanding of these concepts differed significantly from the intended goals of the course. So we presented a fixed number of concepts, but asked them to verbalize the relations between the concepts. We had in mind that misconceptions might be revealed thus. Moreover, we were interested in the number of relations between different concepts, because it may be regarded as a measure of importance for a student. Nonetheless, we had to assess every defined relation. We determined not only whether it was true or false, but also whether it was a proposition with general or specific contents. The following concepts were used:

> program, input, repetition, call, condition, loop, recursion, name of a program, stop condition, program line.
>
> running a program, repeat, subprogram, if...then, a procedure calls itself, nesting.

The concepts may be divided into four groups, and every concept belongs to at least one of them. We will here present our classification, which was a basis for the choice of objects.

There are concepts which are used by students and teachers during progamming instruction and which may be referred to as part of the fundamental vocabulary. These concepts are *program, program line, name of a program, running a program, subprogram*. In a second group we included concepts which reflect structural aspects expressed in natural language (but not necessarily colloquial language). These concepts are *repetition, loop, nesting, recursion, condition*. Another four concepts may be classified as indicating actions with respect to a given program. These concepts are *input, call, a procedure calls* *itself,* and (at least in some respect) *stop condition*. The last group includes the programming language commands *repeat* and *if...then*.

The concepts were presented to a seventh grader and to an eighth grader after six months of LOGO programming instruction. The subjects were JAN, aged 14, a boy who was pretty successful in programming but lacked extra access to a computer, and TOM, also aged 14, a student who performed on a medium level during the course, but was able to work on a computer of his own. The assessment of declarative knowledge was only part of a number of interviews, which also included the assessment of procedural knowledge. In this respect, JAN may be regarded as problem solver using recursive strategies, whereas TOM prefers iterative solutions (HAUSSMANN, 1987; HAUSSMANN & REISS, 1989). The following table shows the number of the established relations.

Table 1: Individual declarative knowledge of two students

CONCEPT	IS OBJECT OF A PROPOSITION		IS SUBJECT OF A PROPOSITION		TOTAL	
	TOM	JAN	TOM	JAN	TOM	JAN
PROGRAM	4	4	3	5	7	9
INPUT	4	2	4	7	8	9
REPETITION	6	3	1	6	7	9
CALL	9	3	1	1	10	4
CONDITION	4	3	1	2	5	5
LOOP	4	2	3	3	7	5
RECURSION	-	1	-	.	-	2
NAME OF A PROGRAM	2	2	4	1	6	3
STOP CONDITION	3	2	2	2	5	4
PROGRAM LINE	2	9	4	1	6	10
RUNNING A PROGRAM	0	2	5	1	5	3
REPEAT	0	1	5	1	5	2
SUBPROGRAM	0	3	5	2	5	5
IF...THEN	-	0	-	1	-	1
A PROCEDURE CALLS ITSELF	4	2	0	1	4	3
NESTING	1	1	5	3	6	4

Results

TOM regards *call* as the most central concept, that is the node with the maximum number of ten connections. There are nine edges with direction to *call* so this concept is used as an object of a proposition in order to clarify others. But actually the propositions show that *call* is regarded on a very general level. It is connected to other concepts only by relations like *is*, *may be*, or *may include*. During the Logo course we used *call* when a procedure was initiated by another. Tom's understanding of the concept includes this point of view as well as a number of vague or even faulty conceptions.

Tom*'s* concept of *"call"*
repeat is a *call*
call may be *repetition*
nesting is a *call*
stop condition is a *call*
input may be a *call*
in *program line* may be included a *call*
name of a program is input for the *call* of a program
subprogram may be *call*
program is *call* for a computer so that it knows what to do
condition is *call* at the same time

The interpretation on a very general level is also true for Jan's main concept *program line*. There are ten relations defined to other concepts but most of these connections use *program line* as an object. The propositions use *has*, *may be*, and *needs* as verbs.

Jan*'s* concept of *"program line"*
a procedure calls itself has *program line*
nesting has a *program line* of its own
subprogram has *program line*
program line may be loop
program needs *program line*
a repeat *loop* has two *program lines*
stop condition has *program line*
condition has *program line*
input has *program line*
input may be *program line*

Another important concept for Tom is *input* which is, as *is call*, a concept indicating a possible action. This concept is used as subject and as object of a proposition four times each. But once more Tom's relations express only vague connections using verbs like *may be* or *is*. So he states that *input may be a program*, *input may be a condition*, or *input may be a call*. The examples show that Tom's concept of *input* is not that of an *input to a procedure* but rather a concept which stands for *touching a key on the keyboard*.

Tom*'s* concept of *"input"*
input may be *program*
input may be *condition*
input may be *call*
nesting needs *input* for reaching the goal
input may be *loop*
name of a program is *input*

subprogram is *input*
running a program needs *input*

Looking at Jan's concept of *input*, there are similar findings. The meaning of *input* is not restricted to an input to a procedure.

Jan's concept of "*input*"
input may be *name of a program*
if *a procedure calls itself* by the name of the program there is an *input*
input may be *subprogram*
different *inputs* may be represented in a *nesting*
input has *program line*
input may be *program line*
loop needs *input*
program needs *input*
input may be *call*

The concept map of Jan reveals *repetition* as another important concept with respect to the number of relations. This concept belongs to those describing structural aspects in natural language. The concept map indicates that Jan has a pretty elaborated understanding of the structural aspect of repetition. It is contrasted with Tom's concept of *repetition*.

Jan's concept of *"repetition"*
repetition with program name loop causes *a procedure calls itself*
repeat and *repetition* are identical
repetition is condition that program needs *loop*
a *repetition* loop has two *program lines*
loop may be *repetition*
repetition and *recursion* are identical
an infinite *repetition* loop needs a *stop condition*

Tom's concept of *"repetition"*
repeat is *repetition*
in a *subprogram* there may be a *repetition*
running a program may cause a *repetition*
repetition causes *a procedure calls itself* after input
nesting may cause *repetition*
loop is a *repetition*
call may be *repetition*

The concept of *repetition* shows important differences between Tom and Jan. Whereas Tom's propositions use mostly vague verbs like *is* or *may be*, Jan's understanding of repetition is meaningful and initiates concrete operations. Nonetheless, the proposition *repetition and recursion are identical* reveals that there are still misconceptions.

Conclusions

Concept mapping may be regarded as a means for assessing declarative knowledge. In particular, in our study we were able to identify misconceptions and partial misconceptions with respect to knowledge on control structures. The results indicate that even after some months of LOGO instruction relevant fundamental concepts may not be learned in an appropriate way. In the classroom, it might be necessary to clarify the use of a specific concept prior to the solution of a programming problem. Moreover, the results indicate that the number of defined relations cannot be a measure for concept understanding. But they show as well that concept mapping might be an adequate instrument for diagnosing concepts and misconceptions. Every student has a subjective theory about the domain taught in classroom which may reflect either an intense knowledge or only a vague idea about the subject. Concept maps are helpful for understanding the students misconceptions in the domain in order to optimize instruction quality.

References

Anderson, J.R. (1983). *The architecture of cognition.* Cambridge: Harvard University Press.

Ballstaedt, S.P. & Mandl, H. (1985). *Diagnosis of knowledge structures in text learning* (Forschungsbericht, 37). Tubingen: Deutsches Institut fur Fernstudien.

Feldmann, K. (1979). MEAP - eine Methode zur Erfassung der Alltagstheorie von Professionellen. In B. Schön & K. Hurrelmann (Eds.), *Handbuch der Curriculumforschung* (pp. 105-122).

Haussmann, K. (1987). Mental representation of recursive structures. In J. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education, Volume III* (pp. 57-62). Montreal: University of Montreal.

Haussmann, K. & Reiss, M. (1989). Strategien bei Problemen mit rekursiver Lösung. Eine prozeßorientierte Analyse rekursiven Denkens. *Journal fur Mathematikdidaktik*.

Lesgold, A. (1988). Intelligenter computerunterstutzter Unterricht. In H. Mandl & H. Spada, *Wissenspsychologie* (pp. 554-569). Munchen: Psychologie Verlags Union.

Miller, G.A. (1956). The magical number seven, plus or minus two: some limits on our capacity for processing information. *Psychological Review, 63*, 81-97.

Novak, J.D., Gowin, D.B., & Johansen, G.T. (1983). The use of concept mapping and knowledge vee mapping with junior high school students. *Science Education, 67*, 625-645.

Reiss, M. (1987). Begriffliches Verstandnis und Aufgabenlösen im Bereich der Proportionen. *Beitrage zum Mathematikunterricht*, 266-269.

Scheele, B. & Groeben, N. (1984). *Die Heidelberger Struktur-Lege-Technik (SLT) - eine Dialog-Konsens-Methode zur Erhebung Subjektiver Theorien mittlerer Reichweite*. Weinheim: Beltz.

Shapiro, S.C. (1987). *Encyclopedia of artificial intelligence.* New York: Wiley.

Tack, W. (1987). Ziele und Methoden der Wissensreprasentation. *Sprache & Kognition, 3*, 150-163.

Tergan, S.O. (1988). Qualitative Wissensdiagnose - Methodologische Grundlagen. In H. Mandl & H. Spada, *Wissenspsychologie* (pp. 400-422). Munchen: Psychologie Verlags Union.

Wender, K.F. (1988). Semantische Netze als Bestandteil gedachtnispsychologischer Theorien. In H. Mandl & H. Spada, *Wissenspsychologie* (pp. 55-73). Munchen: Psychologie Verlags Union.

Winograd, T. (1975). Frame representations and the declarative/procedural controversy. In D. Bobrow & E. Collins (Eds.), *Representation and understanding*. New York: Academic Press.

A CONCEPTUAL ANALYSIS OF THE NOTION OF LENGTH AND ITS MEASURE

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The objective of this study is to establish a descriptive framework for the constuction of the notion of length and its measure by children in primary school. This is achieved by using a two-tier model of understanding. The first one involves three levels of understanding of the concept of length, length being perceived here in a global sense, as an unmeasured physical entity. The second tier describes three components of understanding of the emerging mathematical concept, that of the measure of length. Length is then viewed numerically, in terms of quantification. These different aspects of understanding are described through situations that correspond to appropriate criteria.

The notion of measure in one and two dimensions is presently the subject of a research projet carried out at the Université de Sherbrooke (Héraud,1987). The objective of the present paper is to present for discussion the conceptual analysis which is the theoretical basis of an investigation of the child's construction of the concept of length and its measure.

In the last ten years, several researchers have studied the problems encountered by children in the learning of measurement in general and more particularly, the measurement of length. Based on the results of the second assessment of the National Assessment of Educational Progress (NAEP), Carpenter et al. (1980) have pointed out that, at the end of primary school, many children had but a very superficial understanding of the basic measu[illegible]pts. For instance, regarding the use of a ruler, if the measuring segment started at 1 and not at 0 on the ruler, only 19% of the 9-year-olds were then able to provide a correct answer. Hart (1981) has shown that even in secondary school, there were many students who still had problems with the conservation of length with respect to a simple displacement due to the fact that they were focusing on the end points of the segments and not on their length. Bessot and Eberhard (1983) have tried, with children aged 7 and 8, to get a closer assessment of the difficulties involved in measuring length, such as those found in identifying the proper ruler marks to determine the length of an object when the initial end point is not lined up with 0.

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Other problems related to the learning of measurement have been investigated. In a study of first graders, Hiebert (1984) has brought out the difficulties related to the use of units of different size. In general, children were unable to recognize the inverse relation between the size of the units and the resulting measurement number. In comparing measures, they only took into account the number of units used but not their relative size. Thus, it seems that an understanding of the notion of unit is at the heart of the problem of understanding the notion of measure.

More recently, Boulton-Lewis (1987) has tried to assess the development of the concept of the measure of length by determining a hierarchy of the tasks handled by children aged from 3 to 7 years. The present study has the same orientation. Its aim is not only to find all the difficulties related to the learning of the measure of length, but also to order them in a sequence that may enable us to get a better grasp of the child's construction of this concept.

MODEL USED IN THE ANALYSIS

In order to achieve our objectives, we plan to perform a conceptual analysis of the notion of length and its associated measure that will enable us to determine the main steps in the construction processes used by the learners. To achieve this, we will use a model developed by Herscovics & Bergeron (1988) which suggests that the construction of some mathematical concepts can be described within a framework of a two-tier model of understanding, the first tier describing the understanding of preliminary physical concepts, and the second tier identifying the understanding of the emerging mathematical concept.

In this model, the **understanding of preliminary physical concepts** involves three levels of understanding:

> **intuitive understanding** which refers to a global perception of the notion at hand; it results from a type of thinking based essentially on visual perception; it provides rough non-numerical approximations;
>
> **procedural understanding** refers to the acquisition of logico-physical procedures (dealing with physical objects) which the learners can relate to their intuitive knowledge and use appropriately;
>
> **logico-physical abstraction** refers to the construction of logico-physical invariants, the reversibility and composition of logico-physical transformations and generalizations about them.

The understanding of the **emerging mathematical concept** can be described in terms of three components of understanding:

procedural understanding refers to the acquisition of explicit logico-mathematical procedures which the learner can relate to the underlying preliminary physical concepts and use appropriately;

logico-mathematical abstraction refers to the construction of logico-mathematical invariants together with the relevant logico-physical invariants, the reversibility and composition of logico-mathematical transformations and operations, and their generalization;

formalization refers to its usual interpretations, that of axiomatization and formal proof which at the elementary level could be viewed as the discovery of axioms and the elaboration of logical mathematical justifications. Two additional meanings are assigned to formalization: that of enclosing a mathematical notion into a formal definition, and that of using mathematical symbolization for notions for which prior procedural understanding or abstraction already exist to some degree.

This model suggests a distinction between on one hand, logico-physical understanding which results from thinking about procedures applied to physical objects and about spatio-physical transformations of these objects, and on the other hand, logico-mathematical understanding which results from thinking about procedures and transformations dealing with mathematical objects. We will now use this model to describe the primary schoolchildren's understanding of length and of its measure.

THE UNDERSTANDING OF PRELIMINARY PHYSICAL CONCEPTS

This first classification leads us to distinguish between length and its measure. At this first tier, we consider length as a still unmeasured one-dimensional physical magnitude. We now examine the different levels of understanding that can be determined according to the above model by specifying appropriate criteria.

Intuitive understanding. At this initial level the children's judgments are based on a visual frame of reference. They can thus state that a given object is long or short according to their visual perception of it. This distinction is closely linked to their knowledge of "little" or "a lot" and related to concrete situations of their daily life. For instance, this is how they will judge the amount in a strip of licorice they have, as a function of the length associated with this quantity.

At this intuitive level children are also capable of estimating the respective lengths of two objects by simple visual estimation. They can thus perform direct comparisons of the type "This object is longer (or shorter) than that one", without having to pick up the objects and putting them side by side: they rely on their visual perception.

Procedural understanding. Moving beyond visual estimation, the children will feel the need to use a logico-physical procedure that will guarantee the accuracy of their prior judgment, especially when the difference between the lengths is very small. The simplest procedure consists in aligning side by side the two objects to be compared and verifying which one extends beyond the other. It should be noted that in this procedure there is no need for any quantification. We remain at a logico-physical level where the children are using what might be called a comparative measure in the sense that they use one object to compare it with another one in order to estimate their relative lengths.

This primitive procedure for direct comparison between two objects of different lengths can then be generalized to the seriation of a whole set of objects. For instance, one could envisage a situation in which the children are given a set of rods of different lengths but already ordered, and from which one of the rods has been removed. They would then have to replace this rod by direct comparison with those already laid out. A more difficult task would consist in giving them a whole unordered set of rods of different lengths and asking them to arrange them in an appropriate order. To do this, they could of course proceed by visual estimation for the obvious cases; but, if the difference in length between some rods is very small, they then have to compare them two at a time to order them relatively to each other.

Logico-physical abstraction. To identify abstraction in the logico-physical sense, we can use as criterion the perception that children may have of the invariance of the length of an object with respect to various figural transformations. If they can overcome the disequilibrium induced by the erroneous information received from their visual perception, this can then be taken as evidence of a certain degree of abstraction.

It is at this level that one can use some well-known tasks developed by Piaget et al. (1948/1973) on the conservation of length, such as the one on the invariance of length with respect to unidirectional **displacement**. For instance, one can place two identical straws one below the other and then perform a very slight translation on one of them: do the two straws have the same length? A variation of this task might verify if children believe the straws are still the same length when they are placed next to each other, and part of one being hidden in front of them. In this case, one could call it the invariance with respect to the **visibility** of the object.

Other forms of invariance can be envisaged such as the invariance with respect to the **orientation** of an object. Thus, taking the two identical straws and placing one perpendicular or oblique to the other, one could verify how it affects the child's perception of length. Has it remained the same or has it changed? Another form of invariance is related to the **fragmentation** of the object. For instance, if a straw is cut up into several parts, is its length conserved by the child? Even a more complex task can be designed that might involve both the disposition and the fragmentation of the object. Given two rods, one of them having been split up, the parts can be arranged along a non-rectilinear "path".

UNDERSTANDING THE EMERGING MATHEMATICAL CONCEPTS

At this second tier, we extricate length from its purely physical aspect in order to consider it under its quantifiable, numerical aspect, that is, in the context of its measure. The Herscovics and Bergeron model (1988) described before, enables us to identify three components, and these will now be examined in greater detail.

Procedural understanding. As soon as measuring length is involved, one must necessarily bring in the notion of **unit.** A simple way of measuring the length of an object is to proceed by the iteration of the unit. But while this might appear to be a very simple task, the children are faced with many problems. For instance, can one use indiscriminately several kinds of units or must they all be the same length? How should they then be arranged: can they be partially overlapping or can there be gaps between them? Must one have as many units as needed to cover the length of the object to be measured or can one do with fewer units or even just one unit? All these questions are non trivial for the children and the answers they find will lead them to discover the meaning of unit and will also bring them to use progressively more involved procedures. Initially, they become aware of the importance of using **identical** units which they then learn to place carefully one after the other in order to find the length of an object. Then, in a more sophisticated procedure, they learn to use increasingly fewer units. Finally, they proceed by genuine **iteration** using one single unit that serves as a measuring standard.

Logico-mathematical abstraction. One of the first ways to identify some logico-mathematical abstraction of the measure of length is to examine whether or not children are capable of judging the invariance of this measure with respect to different figural transformations, in situations where a **measuring standard** is known and used. In this sense, the various tasks used to evaluate the invariance of length at the first tier, at the level of logico-physical abstraction, can now be repeated here, but by adding to them this new dimension provided by units.

Moreover, at this level, the child should be able to grasp the links between apparently contradictory aspects of length and its associated measure. For instance, children can find themselves in the following conflictual situations: on one hand, the length of an object taken as a physical entity is invariant; on the other hand, its measure can be expressed in different ways depending on the choice of the measuring standard! Thus, the child must discover that regardless of the unit chosen, the size of the object remains the same, even if its measure varies. The resolution of this conflict is at the very heart of the processes involved in understanding measure.

Another important relation that the child must establish is that of the inverse relation existing between the numerical measure and the size of the unit: the smaller the unit, the larger the numerical measure of the object. Another problem related to the notion of approximation and stemming from it, concerns non-integral measures. In this case, the child must be able to choose an appropriate unit in terms of the desired degree of accuracy.

Formalization. This last component of understanding can cover many different aspects. For instance, it may involve the computation of a measure using **conventional units** and the use of their symbolic representation. It is only at this level that their utilization acquires its true meaning, when the child can use them appropriately and understands ratios existing between the different units.

It is also at this level that one can include the problems related to the introduction of the **ruler** as a measuring instrument; its use involves the formalization of notions acquired previously. The rational use of such an instrument is not as simple as one may believe and it requires the prior resolution of several problems by the child. Among these, one can mention the need to discover the link between the various marks appearing on the ruler and the units associated with them. Another example is the distinction that must be made between the coordinates of the extremities of an object on a scale and the real length of the object.

CONCLUSIONS

The use of Herscovics and Bergeron's Extended Model of Understanding (1988) has enabled us to establish a conceptual framework allowing for a better grasp of the various stages that can lead to the children's construction of the concept of length and its measure. One of the great advantages of this model is that it indicates how a mathematical concept rests on the understanding of preliminary physical concepts. Thus, in the present case, it enables us to distinguish clearly between the concept of length, which is part of the logico-physical domain, and the measure of length, which is part of the logico-mathematical domain. It is not a distinction that relates to the mental processes, but rather to the objects to which these processes are applied. This model should not be perceived as linear, for it allows the overlapping of components of the second tier with those of the first tier (e.g., a child may be using some measuring processes without having reached the level of logico-physical abstraction). It has also the merit of establishing a classification involving two tiers that enable us to easily identify different stakes in the construction of a conceptual scheme and to get a better grasp of its various stages.

It also allows us to get a better overview of the difficulties children encounter in such a construction. These problems acquire a new meaning in the sense that, with this model, one can get a better grasp of their root causes and thus provide a better explanation. For instance, the difficulties that children face in learning to use a ruler might be reduced if they were not asked to utilize such an instrument prematurely.

As can be seen, the interest in this model is not just theoretical. For instance, at the pedagogical level, it strongly suggests that thelearning of length should be based on concrete activities related to the child's physical environment as a basis for the mathematization process. Moreover, it enables us to conceive many complementary tasks related to the construction of this concept. When these activities are developed in correspondence with the different aspects of understanding that we have identified, they should allow us to establish a progression in the construction of length and its measure, progression that would have a better basis and be more pertinent than the one found with a traditional approach.

REFERENCES

Bessot, A., & Eberhard, M. (1983). Une approche didactique des problèmes de la mesure. **Recherches en didactique des mathématiques, 4**, 293-324.

Boulton-Lewis, G. M. (1987). Recent cognitive theories applied to sequential length measuring knowledge in young children. **British Journal of Educational Psychology, 57**, 330-342.

Carpenter, T. P., Kepner, H. S., Corbitt, M. K., Lindquist, M. M., & Reys R. E. (1980). Results and Implications of the Second NAEP Mathematics Assessments: Elementary School. **Arithmetic Teacher, 27** (8), 10-12, 44-47.

Hart, K. M. (Ed.). (1981). **Children's understanding of mathematics: 11-16.** London: John Murray.

Héraud, B., (1987). Conceptions of area units by 8-9 years old children. In J.C. Bergeron, N. Herscovics, & C. Kieran (Eds.), **Proceedings of the Eleventh International Conference of the Psychology of Mathematics education, 3**, 299-304.

Herscovics, N., & Bergeron, J. C. (1988). An extended model of understanding. In M. J. Behr, C. B. Lacampagene, & M. M. Wheeler (Eds.), **Proceedings of PME-NA-10**, 15-22, Dekalb, Illinois.

Hiebert, J. (1984).Why do some children have trouble learning measurement concepts? **Arithmetic Teacher, 31** (7), 19-24.

Piaget, J., Inhelder, B., & Szeminska, A. (1948/1973). **La géométrie spontanée de l'enfant**. Paris: Presses Universitaires de France.

THE KINDERGARTNERS' UNDERSTANDING OF CARDINAL NUMBER: AN INTERNATIONAL STUDY

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This paper reports the results of an international study on the kindergartners' understanding of cardinal number. This understanding has been investigated through various tasks determining if the children perceive the uniqueness of the cardinality of a set, and also its invariance with respect to the direction used in counting a row of objects. Four other tasks were used to assess the child's perception of the invariance of the plurality of a set as well as the invariance of the quotity of the set under various irrelevant spatio-physical transformations. Data on samples of kindergartners interviewed in Montreal, Paris, and Cambridge, Mass. are reported in this communication.

The kindergartners numerical knowledge is of prime interest to both teacher and researcher. On one hand, teachers have to know the extent and depth of the cognitive baggage the children bring with them to primary school in order to establish some cognitive continuity between their experience and the planned arithmetic instruction. On the other hand, for researchers this age group is of particular interest since they can literally witness a cognitive explosion taking place under their own eyes.

Our investigation of the kindergartners' numerical knowledge is now in its fifth year and our results reflect new approaches both at the theoretical level and at the methodological level. At the theoretical level, our research has started with an epistemological analysis of the number concept. This provided us with an overview enabling us to perceive number as a conceptual scheme, that is as a network of related knowledge together with the "problem-situations" in which it can be used. Regarding our methodology, we have adopted the clinical approach used in case studies but have tried to go beyond a few individual cases by using larger samples, averaging thirty odd children, in order to identify likely patterns of thinking.

The term 'epistemological analysis' refers to the analysis of a conceptual scheme along likely patterns of construction by the learner. In our work we have performed such conceptual analyses by applying a two-tiered model of understanding (Herscovics & Bergeron, 1988a), the first tier describing the understanding of the preliminary physical concepts, and the second tier identifying the understanding of the emerging mathematical concept.

Applying this model to the number concept we have identified the notion of **plurality**, that is, the distinction between one and many, and the notion of **position** of an element in an ordered set as two preliminary physical concepts

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(Bergeron & Herscovics,1988, Herscovics & Berge[illegible]8b). Defining number as a **measure of plurality** and also as a **measure of position**, we could identify these as the emerging numerical concepts.

Using the above analysis we have developed a sequence of about forty tasks aimed at uncovering the child's numerical knowledge in about three or four interviews lasting on average 30 minutes each. These interviews were carried out in each of three cities (Montreal, Paris & Cambridge, Mass.) with 30 average children selected by the school authorities. The choice of cities was based on our desire to compare samples with language affinities (Montreal & Paris) and samples with cultural affinities (Montreal & Cambridge).

Among the four Montreal schools two were located in higher socio-economic suburbs whereas the other two were in a lower socio-economic neighborhoods (lower middle-class & working class). The Martin Luther King Jr Elementary School in Cambridge provided us with children in four different classes, two of these being considered as regular kindergarten classes, the other two following an activity-based mathematics program for early childhood education based on Mary Baratta-Lorton's Mathematics Their Way (198?). Both samples from Paris and Cambridge originated from schools located in lower socio-economic neighborhoods.

Two other variables beyond our control were the age difference between the samples and the date of the interviews: The 29 Parisian children had an average age of 5:8 and were interviewed between the last week of February and the first week of April 1988; the 30 Cambridge kindergartners had an average age of 5:10 and were interviewed between the end of April and the beginning of June 1988; the 32 Montreal children had an average age of 6:2 and were interviewed between the end of April and the beginning of June 1988.

The present paper will cover the logico-mathematical abstraction of cardinal number. A companion paper in these Proceedings deals with the the abstraction of ordinal number (Bergeron & Hersovics, The kindergartners' understanding of ordinal number).

In view of the first part of our definition of number as a measure of plurality , the logico-mathematical abstraction of number must reflect both the **invariance of plurality** and the **invariance of its measure** with respect to irrelevant spatio-physical transformations, leading to the abstraction of cardinal number. We now describe the various tasks designed to assess the children's understanding of these notions.

Uniqueness of the cardinality of a set. Ginsburg (1977) has pointed out that some young children can enumerate a given set several times and obtain different results without necessarily developing any sense of contradiction. The kindergartners' perception of the uniqueness of the cardinality of a set, as measured by their enumeration, was evaluated by asking each child how many cubes were in a given set (12). After these had been counted, the interviewer told the following story: "When I asked another little friend how many cubes there were here, he told me there were eleven. Do you think that you are right, or that he is right, or that both of you are right?". In each of the three cities only one child in each sample (n=29;30;32) thought that both answers could be right. This indicates that by the time children finish kindergartnen, they are aware of the uniqueness of the cardinality of a set. Most of the children in Montreal and Cambridge went about counting the cubes a second time and then immediately affirmed that they were right and the friend was wrong. It is interesting to note that many Parisian children interviewed first responded by claiming that both answers were right. However, further questioning revealed that they did not really believe the answer they had given for they too ended up recounting the cubes and claiming their answer as the right one. A discussion with their teachers indicates that their initial response could be explained by the emphasis of "getting along" that is stressed in French kindergartens.

Invariance of cardinality with respect to the direction of the count . Another aspect of the cardinality of a set is its invariance with respect to the order of enumeration of the objects (Piaget,1973; Gelman & Gallistel,1978). A simpler problem involves the invariance of the cardinality with respect to the direction used in counting up a row of cubes. Earlier research had shown that 77% of kindergartners were aware of this invariance (Herscovics et al.1986). The results obtained in the present study were somewhat better. The children's perception was ascertained by aligning in a row 12 identical cubes in front of a child who then had to find how many there were. Following the enumeration, the subject was asked "If you start counting from here (indicating the end point of the initial count), how many will you find?". In case the child counted the row a second time the interviewer asked "Did you need to count them that way? (indicating the second direction)?". Notwithstanding the given answer, another set of 10 cubes was aligned and the task was repeated in order to verify if the the child would count again in the second direction. The reason for repeating the task was that for many students the words "How many?" trigger a counting response and for others their second count is not so much used to verify the number of elements but more as a demonstration intended for the interviewer. The following table shows the results obtained:

City	Succeeded on first try	Succeeded on second try	Did not succeed
Cambridge			
Regular classes (n=14)	12	1	1
Lorton classes (n=16)	14	2	0
Totals	26 (86.7%)	3 (10%)	1(3.3%)
Paris (n=29)	20 (69.0%)	5 (17.2%)	4 (13.8%)
Montreal			
Higher socio-econ (n=16)	13	3	0
Lower socio-econ (n=16)	9	4	3
Totals	22 (68.8%)	7 (21.8%)	3 (9.4%)

It should be noted that the overall success rate is fairly high in the three samples (96.7%, 86.2%, and 90.6% respectively). The Parisian and Montreal children seem to have very similar success rates whereas the Cambridge sample indicates that all but one child perceive this invariance.

INVARIANCE OF PLURALITY AND INVARIANCE OF QUOTITY

Over twenty years ago, Piaget's collaborators Greco (1962) felt the need to distinguish between the children's conception of plurality and the meaning they attach to enumeration. They modified the original conservation task involving two equal rows of chips by asking the children to count one of the rows before stretching the other one; they then asked how many chips were in the elongated row while screening it from view. Those who could answer the question were said to **conserve quotity**. Greco found that many five-year-olds claimed that there were seven chips in each row but that the elongated row had more. Thus, these subjects **conserved quotity without conserving plurality**. For these children, to conserve quotity simply meant that they could maintain the numerical label associated with the elongated row, but their count was not yet a measure of plurality, since they thought that the plurality had changed. It is only when both plurality and quotity are conserved, when both invariances are perceived, that number can become a measure of plurality. At that stage, one can claim that the child has achieved a logico-mathematical abstraction of cardinal number. Of course, the Piaget and the Greco tasks are not the only ones which can be used to assess abstraction of cardinal number. We have theirs and designed three other tasks for our assessment.

Invariance with respect to the elongation of a row. During the first interview each child was presented with a row of 11 identical cubes and was told : "Here is a row of cubes. Look, I'm going to stretch it out....Now, do you think that there are more cubes...les cubes... or the same number as before I stretched the row?". In the third interview the child was presented with the same row of cubes, but this time was asked right at the beginning "Can you tell me how many cubes we have?". After the count, the row was stretched out and the interviewer asked: "Now, without counting, can you tell me how many cubes are in the row?" (while screening off the row from the child's view with a forearm or two hands to prevent any counting). The following table provides the data for these tasks:

City	Invariance of plurality	Invariance of quotity	Invariance of both
Cambridge			
Regular classes (n=14)	13 (92.9%)	11 (78.6%)	10 (71.4%)
Lorton classes (n=16)	16 (100%)	15 (93.8%)	15 (93.8%)
Totals	29 (96.7%)	26 (86.7%)	25 (83.3%)
Paris (n=29)	16 (55.2%)	24 (82.8%)	14 (48.3%)
Montreal			
Higher socio-econ (n=16)	14 (87.5%)	14 (87.5%)	13 (81.3%)
Lower socio-econ (n=16)	12 (75.0%)	16 (100%)	12 (75.0%)
Totals	26 (81.3%)	30 (93.8%)	25 (78.1%)

The data indicate that on the conservation of quotity there is a remarkably high rate of success in the three samples. On the invariance of plurality , the average for the Parisian children is much lower than for the other two groups. The fact that they were 2 months younger than the Cambridge children and 4 months younger than the Montreal ones may account for some of this difference, as well as the fact that they were interviewed two months earlier in the school year.

Invariance with respect to the dispersion of a set. Tasks analogous to the preceding ones were used to assess the children's perception of the invariance of a set of 9 identical cubes laid out randomly and then spread out in front of them. The following table shows the results obtained:

City	Invariance of plurality	Invariance of quotity	Invariance of both
Cambridge			
Regular classes (n=14)	11 (78.6%)	10 (71.4%)	8 (57.1%)
Lorton classes (n=16)	15 (93.8%)	16 (100%)	15 (93.8%)
Totals	26 (86.7%)	26 (86.7%)	23 (76.7%)
Paris (n=29)	19 (65.5%)	25 (86.2%)	19 (65.5%)
Montreal			
Higher socio-econ (n=16)	14 (87.5%)	16 (100%)	14 (87.5%)
Lower socio-econ (n=16)	11 (68.8%)	13 (81.3%)	10 (62.5%)
Totals	25 (78.1%)	29 (90.6%)	24 (75.0%)

As in the previous set of tasks the success rate on the invariance of quotity is high for the three groups. All but one child from the two Lorton classes have also acquired the invariance of plurality. What is strikingly similar is the result obtained on plurality in the regular Cambridge classes, the Parisian children, and the Montreal classes in lower socio-economic neighborhoods (78.6%, 65.5%, and 68.8% respectively).

Piagetian tasks. The third set of tasks used to assess the invariance of cardinality were the classical Piagetian test on the conservation of plurality and the Greco modification mentioned earlier. The following table shows the results obtained:

City	Invariance of plurality	Invariance of quotity	Invariance of both
Cambridge			
Regular classes (n=14)	8 (57.1%)	12 (85.7%)	8 (57.1%)
Lorton classes (n=16)	16 (100%)	16 (100%)	16 (100%)
Totals	24 (80.0%)	28 (93.3%)	24 (80.0%)
Paris (n=29)	7 (24.1%)	21 (72.4%)	7 (24.1%)
Montreal			
Higher socio-econ (n=16)	13 (81.3%)	14 (87.5%)	12 (75.0%)
Lower socio-econ (n=16)	8 (50.0%)	12 (75.0%)	7 (43.8%)
Totals	21 (65.6%)	26 (81.3%)	19 (59.4%)

Results indicate a maximal rate of success among the children following the Barrata-Lorton program. On the invariance of plurality, the sample from the regular Cambridge classes compares with the sample from the two Montreal lower socio-economic neighborhoods. The sample of Parisian children achieves a much lower rate (24.1%). Again, this can be attributed in part to their younger age. However, this result is fairly consistent with their earlier performance on the elongation of a single row, for their success rate there was 20% lower than the lowest results obtained in Montreal (55.2% vs 75.0%).

Invariance with respect to the visibility of the objects. In this last set of tasks on the invariance of cardinality, children were given in the first interview a row of 11 chips glued on a piece of cardboard. They were told: "Here is a large cardboard with little chips glued to it. Look, I'm putting the cardboard in a bag (the interviewer inserting the cardboard in a transparent bag). Good, are all the chips in the bag?". Following confirmation: "Look, I'm putting a plastic strip in the bag (the interviewer inserting a platic strip with an opaque part large enough to cover three chips). And now, are there more chips in the bag, less chips, or the same number as before?" Usually in the second interview, this task was repeated but the children were asked to count up the number of chips before they were inserted in the bag. The following table shows the results obtained:

City	Invariance of plurality	Invariance of quotity	Invariance of both
Cambridge			
Regular classes (n=14)	1 (7.1%)	10 (71.4%)	1 (7.1%)
Lorton classes (n=16)	5 (31.3%)	14 (87.5%)	5 (31.3%)
Totals	6 (20.0%)	24 (80.0%)	6 (20.0%)
Paris (n=29)	8 (27.6%)	6 (20.7%)	1 (3.4%)
Montreal			
Higher socio-econ (n=16)	3 (18.8%)	13 (81.3%)	2 (12.5%)
Lower socio-econ (n=16)	8 (50.0%)	12 (75.0%)	0
Totals	3 (9.4%)	25 (78.1%)	2 (6.3%)

Whereas the results on the invariance of quotity are similar in Cambridge and in Montreal, their discrepancy with those obtained in Paris is hard to explain. But it is the uniformly low results on the invariance of plurality that are most astonishing. They indicate that among most kindergartners, including those in the Lorton program, the visibility of the objects is still primordial. As pointed out by Hermine Sinclair (personal communication), this is not a question of the permanence of the objects which is acquired well before the age of five. Nor is it a question of the enumerability of the partially hidden set as evidenced by the invariance of quotity. Visibility of the objects affects these children's perception of plurality.

BY WAY OF CONCLUSION

In order to have an overview of the children's understanding of cardinal number, the results (in percents) obtained on the various tasks are summarized in the following table, invariance of cardinality signifying the invariance of both plurality and quotity:

Invariance	Cambridge		Paris	Montreal	
	Lorton classes	Regular classes		Lower income	Higher income
Uniqueness of card.	93.8	100	96.6	100	93.8
Inv.wrt direction of count	100.	92.9	86.2	81.3	100
Inv.wrt elongation of row	93.8	71.4	48.3	75.0	81.3
Inv.wrt dispersion of set	93.8	57.1	65.5	62.5	87.5
Inv.wrt Piagetian tests	100	57.1	24.1	43.8	75.0
Inv.wrt visibility of objects	31.3	7.1	3.4	0	12.5

What is most striking about this table is that apart from the Parisian results obtained on tasks involving the elongation of a set, the basic hierarchy is similar in the three samples. By and large, the uniqueness of the cardinality of a set and the invariance with respect to the direction of the count seem to be achieved in this age group. The Cambridge and Montreal results on the elongation of a row and on the dispersion of a set are similar in the two regular classes and the two lower income classes. The Piagetian tests are more difficult for both Parisian and Montreal children. The invariance with respect to the visibility of the objects has the lowest rate of success in all groups.

Equally striking is the overall success rate obtained by the children following the Baratta-Lorton program. Clearly, the type of activities that enable the child to reflect about the various properties of number can have a strong impact even at this early age.

Also remarkable is a comparison of the success rates in the three middle columns. Again, if the odd results obtained in Paris on the elongations tasks are ignored, very similar rates are found among the Cambridge children from the regular classes, the Parisian children (who also come from a lower middle class and working class area), and the two Montreal classes situated in comparable neighborhoods.

REFERENCES

Baratta-Lorton, M. (198?), **Mathematics their way**, Addison-Wesley: Reading, Mass.

Bergeron,J.C. & Herscovics,N., (1988), The kindergartners' understanding of discrete quantity, in **Proceedings of PME-XII**, Borbas,A. (ed), Veszprém, Hungary, 162-169

Gelman,R., & Gallistel,C.R., (1978), **The child's understanding of number**, Harvard University Press: Cambridge, Mass.

Ginsburg, H. (1977), **Children's arithmetic**. New York: Van Nostrand.

Greco,P (1962), Quantité et quotité, dans Greco, P. et Morf, A., **Structures numériques élémentaires**, Vol. XIII – Études d'épistémologie génétique, sous la direction de Piaget, J. Presses Universitaires de France : Paris

Herscovics,N., & Bergeron, J.C. (1988a). An extended model of understanding, in **Proceedings of PME-NA-10**, Behr,M.J., Lacampagne,C.B., & Wheeler, M.M. (eds), Northern Illinois University, DeKalb, Illinois, 15-22

Herscovics,N., & Bergeron,J.C. (1988b). The kindergartners' understanding of the notion of rank, in **Proceedings of PME-XII**, Borbas,A. (ed), Veszprém, Hungary, 385-392

Herscovics,N., Bergeron,J.C., & Bergeron,A., (1986). The kindergartner's perception of the invariance of number under various transformations, in **Proceedings of PME-NA-8**, Lappan,G. & Even,R. (eds), Michigan State Univ.: East Lansing, Michigan, 28-34

Piaget, J., (1973). Comments on mathematical education, in **Developments of Mathematical Education**, Proceedings of ICME 2, Howson, A.G. (ed), Cambrdige University Press: London, 79-87

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LEARNING ABOUT ISOSCELES TRIANGLES

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As a part of a computer based geometry environment, a procedure for an isosceles triangle, quantified by base and base angle, was given as an object to investigate. The paper looks at the evolution of the children's understanding of this geometric object and of the relationships among its components.

The activities involving isosceles triangles were part of a year long computer-based geometry project. The project was conducted with an entire grade-6 class (12-13 year olds) of average ability, in an elementary school in Montreal. Its general objectives were to provide children with experiences of basic geometric shapes, of their quantifiable components and of the geometrical-numerical relationships that arise out of some geometrical configurations.

The Learning Environment

The sessions spanned 26 weeks and were part of the children's normal school activities. The class split into two groups of 13 children, and each group came to the school's computer lab for a 45-minute session while the other group stayed in the classroom. There were enough computers in the lab for each child to work on a separate machine.

The available programming tools consisted of three geometric objects Rectangles, Circles and (isosceles) Triangles which were given as pre-defined Logo procedures, RECT, CIRCLE and TRI. The procedure RECT needed two positive inputs representing the base and height of the rectangle. CIRCLE's single positive input stood for the diameter. TRI's first input was a positive number representing the base and the second input was a number representing the base angle of the isosceles triangle. (An invalid input for the base angle resulted in the error message "the base angle of an isosceles triangle has to be between 0 and 90".)

Each procedure produced a figure on the screen which was placed in a particular position and orientation relative to the turtle, i.e. a circle with the turtle at its center, a rectangle and a triangle with the turtle at the mid-point of the base and with its heading perpendicular to the baseline. Once the children became familiar with the shapes, the placement of the turtle was changed to a simple marker, as shown in Figure1.

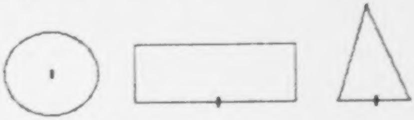
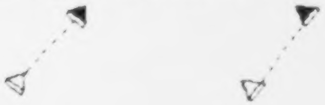


Figure1

There were three commands to manipulate the turtle in the plane: MOVE, SLIDE and TURN, each of which required a single input which could be either negative or positive. MOVE displaced the turtle along the line of its heading, in the same direction if the input was positive and in the opposite direction, if the input was negative. SLIDE displaced the turtle along a line perpendicular to its heading, to its right, if the input was positive, and to its left otherwise (see Figure2). TURN led to a rotation, the input indicating the number of degrees and positive input resulting in a clockwise rotation.



MOVE 50

SLIDE 50

Figure2

Finally, two special numerical procedures were available in conjunction with the isosceles triangle. ALT gave the altitude from the base of the triangle and SIDE gave the length of its equal sides. Both needed the same inputs as TRI, e.g. ALT 100 70 yielded 137, which is the altitude of an isosceles triangle with base 100 and base angle of 70°.

This particular environment resulted in a geometry with transformational and quantitative aspects. The turtle commands of MOVE, SLIDE and TURN, measured in terms of turtle-steps and degrees, allowed for translation and rotation of geometric shapes. The three basic geometric figures were also given as objects with attached measures.

The Isosceles Triangle

The isosceles triangle was introduced in Session 10 (S10). This was not a very familiar object for the children and their notion of angular measure and base angles was almost non-existent. Our choice of parametrizing the isosceles triangle by its base angle rather than its base and side (which are the obvious visible components of the triangle) or base and altitude, was in order to bring the concept of angle into the environment. Furthermore, the base angle serves as a nice example of an invariant of a family of figures, i.e. of similar isosceles triangles. We expected that the children would have difficulties making sense out the second input to the TRI procedure and that their spontaneous conception would be that the second input controls, in some way, the height of the triangle. Many of the tasks that we gave the children were meant to create conflicts with such conception. Also this kind of parametrization conflicted with children's general underlying assumption that if figure A is embedded in figure B, then all the corresponding inputs are larger for figure B (true, for example, if the triangle was parametrized by its base and height)

The numerical procedures ALT and SIDE were introduced by S16. By then, most of the children were aware of the different attributes of an isosceles triangle and of the need to have different kinds of measures in order to solve some of the given tasks in a more analytical way.

Understanding Isosceles Triangles

We examine the children's progressive understanding of the procedure TRI by looking at the work of one child, Jay. His work was rather typical of the behaviour shown by at least 15 children (21 children stayed for the duration of the project).

After the procedure TRI was introduced (and the children were told in an explicit way what the two inputs to TRI signify, though we did not expect that they would make immediate sense of the term 'base angle'), Jay worked on several tasks involving (isoceles) triangles. Among the first was Figure3 involving three similar triangles;

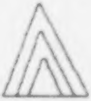


Figure3

In going from the smallest to the largest triangle, Jay initially incremented both the base and the base angle by 10 and after receiving feedback from the screen, he continued by making several adjustments to the second input of TRI.

In S11, Jay worked on Figure4. Again his spontaneous choice was to vary both inputs. After several trial-and-adjustment moves, he did end up with a fixed second input for the three triangles.

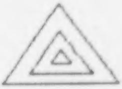


Figure4

His experiences at the end of S11 carried over to S12, and Jay kept a fixed second input of 60 for the two triangles in Figure5.

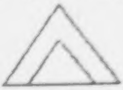


Figure5

However, when he was asked what the 60 stood for, Jay replied "it is the height of the triangle".

It seems that at this point, after three sessions of work on triangles, Jay was entertaining two conflicting ideas. He retained his initial spontaneous conception that the second input to TRI represented height. At the same time he began to recognize that certain configurations involving triangles of different heights involved the same second input. When Jay was confronted with the inconsistency, he once more talked of height, but then corrected himself "no, it [the second input] is angle". Further probing by the observer indicated that though he used the term 'angle', he was not sure which angle was being referred to, and he received some explicit help.

Later in the same session, Jay kept the second input of TRI invariant for the two triangles in Figure6.

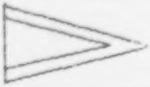


Figure6

On the next task (Figure7), he referred to the second input of for triangle ABC by tracing with his hand the angle at A and saying "it has to be some kind of a low number, less than 65".

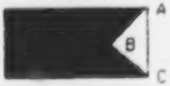


Figure7

S13 started with a blackboard activity and a class discussion. When Figure8 was drawn on the blackboard and a question about the size of the angle was asked, Jay offered 60. He justified his response by drawing a 90-degree angle on the blackboard, then tracing a 30-degree line (Figure9)

Figure8

Figure9

His explanation was: "suppose this is 90, then this 30 is one third, and 60 is twice that" He also was among the pupils who argued that extending the arms of the angle would not change the size of the angle

Jay had shown that, at that stage, he had a good grasp of the notion of angle and of its measure in degrees. His work at the end of S12 also suggested that he began to understand how angle was related to the procedure TRI. Yet, in S14, working once more on a task involving similar triangles (see Figure5), he ended up with inputs of 70 and 65 for the larger and smaller triangles, respectively. Furthermore, 65 was arrived at after several trial-and-adjustments increments, which he carried until the (left) sides of the triangles looked parallel to him.

Jay continued with his conception of TRI even after the numerical procedure ALT to calculate the altitude of a triangle was introduced to the class. For Figure10

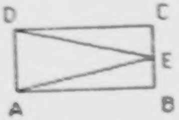


Figure10

Jay started with a 100x50 rectangle (AB = 100, AD = 50) and chose TRI 50 100 for triangle AED, consistent with his interpretation of the second input as height. Since the error message alerted him that the input must be less than 90, he decremented the input by successive trials till he arrived at TRI 50 76, which led to a correct-looking figure. When asked what 76 stood for, Jay replied "the height of the triangle" Once more, he saw no contradiction between his initial (correct) assessment that the altitude of triangle AED must be 100 and his subsequent conclusion that the height was 76

After another discussion with the observer about the procedures TRI and ALT, Jay returned to Figure10, this time starting with triangle AED, then evaluating its altitude and using this value for the dimension of AB

His deliberate use of the procedure ALT on the previous task finally allowed Jay to disentagle the two related notion of height and base angle. From S18 onwards, Jay's work pointed to a consistent interpretation of TRI in terms of base angles, though he experienced some perceptual difficulties in seperating similar amd non-similar triangles. He started to work on tasks which involved relationships such as complementary base angles and complementary and supplementary (turtle) rotations relative to a base angle. His solutions to these tasks showed a definite progression from 'visual' to 'analytical' solution schema (see Hillel & Kieran, 1987). For example, in S18 he worked on Figure11

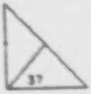


Figure11

After starting with TRI 60 37 for the lower triangle, he figured out correctly that the base of the second triangle is twice the altitude of the first. However, he estimated the second base angle as 40 rather than using complementarity. On the other hand, for Figure12 in S23, he began by choosing 50 and 25 for the base angles of T1 and T2. Asked what he would choose for T3, he answered "I am going to find out how much angle I have used, and take it away from 90"

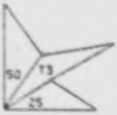


Figure12

All the pupils in the study were interviewed at the end. The interview included six questions on the procedure TRI and on related concepts. Jay's responses to these questions (S24) were precise and correct.

Conclusion

As we have mentioned above, the way Jay's understanding of the procedure TRI evolved was typical of most of the children in the study (Four children hanged on to their initial conception that the second input to TRI parametrized the *side* of the triangle, see Kieran and Hillel (1989), for the complete discussion of the results of the study). From a pedagogical perspective, Jay's behaviour illustrates the persistence of pupils' *initial* conceptualizations and their abilty to accommodate conflictual situations prior to resolving them. What is interesting here is that such 'classical' learning behaviour took place in a very flexible learning environment which allowed ample opportunity to experiment and which provided constant feedback. This reminds us, once more, that pupils need lots of time and experiences before they arrive at an operational understanding of a new concept.

References

Hillel, J. and Kieran, C.,(1987). "Schemas used by 12-year-olds in solving selected Turtle Geometry tasks", *Recherches en Didactique de Mathématiques*, *7.4*

Kieran, C. and Hillel, J.,(1989). "It is tough when you have to make the triangles angle"; Insights from a computer based geometry environment", manuscript submitted for publication.

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CONSTRUCTION OF FUNCTIONS, CONTRADICTION AND PROOF

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Abstract:

The intuitive knowledge with which an individual tackles a problem in mathematics acts in two ways. It may act as a stimulus to progress; or it may be an "anchor" or obstacle, which cannot be changed or removed by means of simple exercises and explanations. It can also lead to a contradictory situation for an individual attempting to solve a mathematical problem and lacking the right tools or strategies to overcome the obstacle. From this perspective, the behaviour of a number of mathematics teachers is analysed by means of a questionnaire on the concept of function and this behaviour is related to the historical development of a mathematical idea. One finding is that the notions of function and continuous function are intuitively assimilated as the single concept "function-continuity".

INTRODUCTION

In what follows we shall be concerned with contradiction and proof. The construction of functions is the vehicle whereby I hope to analyse contradiction and processes in mathematical proof.

It is difficult for an observer or researcher to interpret the intuitive knowledge with which an individual approaches a problem. In some cases, however, it can prove helpful, in interpreting the situation which individuals may find themselves faced with, the study how a mathematical idea has developed over time.

With the foregoing considerations in mind, I undertook a study into the concept of function, both from the historical viewpoint and also from the point of view of a mathematics teacher. For the study I asked 29 mathematics teachers to answer a questionnaire containing 25 questions, all of which were related to the concept of function. The results showed that teachers who gave correct answers generally showed a strong tendency to construct continuous functions, even when the question only specified a function of some kind.

Question 4: Construction of functions.

Construct two functions f_1 and f_2 with the domain ℝ and rank ℝ also, such that

$f(-5) = 2$; $f(0) = 1$; $f(5) = 6$

The results were as follows:

Question	No answer	Incorrect answer	Correct answer continuous function	Correct answer disconti-nuous function
4a	11	8	10	3
4b	14	9	3	2

Of the ten who gave correct answers by constructing continuous functions, seven teachers constructed parabolas for the first function and three made compositions of two semi-straight lines. Of the ten subjects who constructed continuous functions only three were able to construct another continuous function. The remaining seven reached the limit of their ability to construct continuous functions.

The mathematics teachers who took part in the study knew about discontinuous functions. However, their intuitive grasp of the function-continuity concept was an 'anchor' which proved stronger than their awareness of function on its own. The teachers, in other words, had assimilated the concept of function-continuity and could use it when called upon to do so in a natural way; but to make them produce their own ideas and isolate the concept of function, it would have been necessary to say something on the lines of: 'Construct two functions, which need not be continuous, with the following characteristics...'. The history of the concept of function shows us that Euler behaved in a similar way.

Consider another task which the teachers were asked

Question 23:Constructing functions with special properties.

23.Given the property $(f \circ f)(x) = f(f(x)) = 1$ for any $x \in ℝ$.

Construct two different examples, either by means of a graph

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Question 4: Construction of functions.

Construct two functions f_1 and f_2 with the domain $\mathbb{R}$ and rank $\mathbb{R}$ also, such that

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Consider another task which the teachers were asked

Question 23:Constructing functions with special properties.

23.Given the property $(f \circ f)(x) = f(f(x)) = 1$ for any $x \in \mathbb{R}$.

Construct two different examples, either by means of a graph or by making the function explicit, which have this property.

The results of this question were as follows:

Question	No answer	Incorrect answer to 1st item	Correct answer to 1st item	Incorrect answer to 2nd item	Correct answer to 2nd item
23 Making function explicit	4	4	16	7	2
Graph	4	6	17	2	2

This example shows to what extent intuition can have an 'anchor' effect. The effect here was that the teachers, having solved the first part of the question, were prevented from going on to solve the second. The most frequently chosen answer was $f(x)=1$, for any $x \in \mathbb{R}$. Only two teachers were able to break the 'anchor' and separate the intuitive knowledge of function-continuity from the notion of function per se. I would suggest, again, that if the teachers had been told that it did not matter whether the function they produced was continuous or not, the number of correct responses to the second part of the question would have been higher.

In the light of this evidence, we would agree with Fischbein (1973), page 223, when [illegible] assures us that '...intuition cannot be created, eliminated or modified by either explanations or short learning exercises...'. In Fischbein (1982), page 17-18, we find an example related to the theorem in Euclidean Geometry according to which 'the sum of the angles of a triangle is equal to two right angles'. He suggests an intuitive proof of the theorem, and states that 'this representation can be translated directly into a formal proof. The formal proof and the intuitive interpretation are perfectly congruent. Here, intuition is seen as having a direct effect on learning. On the other hand, what we have called the 'anchor' of intuition, which may be related to the notion of 'epistemological obstacle' (Brousseau, 1983), far from being a stimulus to the learning of notions, representations, etc., may actually prevent learning from taking place.

The history of mathematical ideas is a rich source of examples of this process in action. One such can be found in J. Fourier's 'Théorie analytique de la chaleur', Chapter 3 (1812) p.157, in which he assures us that

> '*En general, la limite de la série est alternativement positive et négative: au reste, la convergence n'est point assez rapide pour procurer une approximation facile, mais elle suffit pour la vérité de l'équation.*
>
> *L'équation*
>
> $$y = \cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \ldots$$
>
> *appartient à une ligne qui, ayant x pour abscisse et y pour ordonnée, est composée de droites séparées dont chacune est parallèle à l'axe et égale à la demi-circonférence. Ces parallèles sont placées alternativement au-dessus et au-dessous de l'axe, à la distance π/4, et jointes par des perpendiculaires qui font elles-mêmes partie de la ligne. Pour se former une idée exacte de la nature de cette ligne, il faut supposer que le nombre des termes de la fonction*
>
> $$\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \ldots$$
>
> *reçoit d'abord une valor déterminée.*'

Fourier considered that hence the limiting function was continuous. But, in fact, Fourier's statement does not represent the graph of any function ('*et jointes par des perpendiculaires*'). However, if we consider the intuitive ideas of Fourier, he must have thought that the infinite sum of continuous functions was continuous. His comment on the convergence of the series that 'the convergence is not sufficiently rapid to produce an easy approximation, but it suffices for the truth of the equation' suggests something of the intuitive ideas that Fourier had in relation to this concept.

In Fourier's work, a 'statement in action', to adapt Vergnaud's (1982) expression, can be seen to exist: 'If the terms in the series $u_1, u_2, u_3, \ldots, u_n, u_{n+1}, \ldots$, are functions of a single variable x, which is continuous with respect to this variable in the vicinity of a particular value in which the series is convergent, the sum S of the series is also, in the vicinity of this particular value, a continuous function of x'

The statement is false in the Weierstrass' continuum.

Indeed, until the work of Robinson [1966] (related with Non Standard Analysis), everybody thougt that the above statement was wrong, and Fourier's function was a counter example. But, the statement is a theorem in Non Standard Analysis.

Augustin-Louis Cauchy must surely have been influenced by Fourier's ideas, and it is possible he perceived this 'statement in action' and developed it as a theorem [1821]. In other words, intuition played an important role by generating an argument which led to a false theorem (in Weierstrass' continuum). Cauchy [1853] modified the statement adding another hypothesis to the functions. Did Cauchy think his theorem was wrong ?. Indeed, he wrote (idem p. 31-32) '*Au reste, il est facile de voir comment on doit modifier l'enoncé du théorèm, pour qu'il n'y ait plus lieu à aucune exception. C'est ce que je vais expliquer en peu de mots.*'

Abel N. H., in his article (1826) on binomial series, makes the following statement: '*It seems to me that there are exceptions to Cauchy's theorem*' and proposes, as a counter example:

$$\sin \phi - \frac{1}{2} \sin 2\phi + \frac{1}{3} \sin 3\phi - \dots,$$

In the Weierstrass' continuum context, '*Abel tried to answer the question: What is the safe domain of Cauchy's theorem?*' (Lakatos [1976]). '*It was the mathematician Seidel (1847) who found the error and from this the concept of Uniform Convergence in a predetermined neighborhood of a point was born*' (idem).

Returning to the experiment with 29 teachers in regard to the subject of proof, the results of two other questions in the test were revealing. These two questions had features which were not commonly found in the daily teaching activities of the subjects. In both questions teachers were asked first to say whether a proposition was true or false; if it was true, they were asked to give a proof, and if it was false, to give a counterexample. The questions were intended to 'remove, in the teacher, the classical picture of mathematical development as a steady accumulation of established truths'. As it happened, both propositions in questions 20 and 21 were false, and what was therefore required in each case was either a proof of falsity or a counterexample.

Question 20:

20.Let f and g be two functions of $\mathbb{R}$ in $\mathbb{R}$.

Let $(f(x)) + (g(x)) = 0$ for any $x \in \mathbb{R}$.

Does this implie that $f(x) = 0$ and $g(x) = 0$ for any $x \in \mathbb{R}$?

Yes ☐ No ☐

Explain your answer either by giving an argument in favour of the implication or giving an example of two functions f and g which meet the first condition but where f and g are not functions which annul eath other out in any $x \in \mathbb{R}$.

Explanation:

Question 21:

21.Let f a function of $\mathbb{R}$ in $\mathbb{R}$.

Let $(f \circ f)(x) = f(f(x)) = 0$ for any $x \in \mathbb{R}$.

Does this implie that $f(x) = 0$ and $g(x) = 0$ for any $x \in \mathbb{R}$?

Yes ☐ No ☐

Explain your answer either by giving an argument in favour of the implication or giving an example of a function f which meet the first condition but where f is not the function zero in $\mathbb{R}$.

Explanation:

RESULTS QUESTIONS 20 AND 21	Question No. 20	Question No. 21
No, and gave a counterexample	1	1
No, and gave an erroneous argument	2	0
No, and constructed an 'example', but not a counterexample	0	3
No, and failed to give an explanation	0	1
Totally failed to answer the question	3	16
Yes, and attempted to prove the proposition with unsuitable arguments	17	8
Yes, and gave an unsuitable argument	2	0
Yes, and failed to give an explanation	4	0

Only one teacher solved both questions correctly. In question 21, three teachers constructed a function for which the statement was true as a particular instance, although they said that the proposition was not generally true, their answer was '$f(x) = 0$ for any $x \in \mathbb{R}$'. The argument most favoured by the 17 teachers shown in the table was: 'If either of the two equals zero, let $f(x) = 0 \ \forall \ x \in \mathbb{R}$' or, in other case, though the argument is fundamentally the same: 'The product of two real numbers is zero, if and only if one or both factors equals zero'.

The fact that so many subjects failed to provide any answer to question 21 is significant. The eight teachers who tried to prove the proposition made use of arguments such as: 'Let $f(x) > 0$ for any x, then $f(f(x)) > 0$ for that value of x, similarly for another case'.

The teachers were undoubtedly able to carry out a mathematical proof, but the results show that they had difficulty in applying the general notions of a proof when the proposition was not immersed in the context of direct implication ($A \Rightarrow B$).

In questions 20 and 21, there was a strong tendency to use the direct proof method which made it difficult for subjects to imagine that the proposition might be false and therefore to prove its falsity or to construct a counterexample. All other methods, and even the possibility of a false proposition, were obscured by this prevalence of the direct proof method.

We know that awareness of the presence of a contradiction is not a simple matter (Hitt, 1979). A contradiction may even, as Balacheff (1987) assures us, be an aid to progress, though he points out that there are conditions:

> '*Nous retiendrons les conditions suivantes comme nécessaires à la prise de conscience d'une contradiction:*
>
> *i) existence d'un attendu;*
>
> *ii) possibilité de construire l'affirmation associée à cet attendu et sa négation.*'

In the studies I have mencioned, '*l'existence d'un attendue*' was brought about in a variety of ways. But the main problem arises in '*la possibilité de construire l'affirmation associée à cet attendu et sa négation*'. It is precisely in constructing the negation that the obstacle arises. In this study on the concept of function, I show that the concepts of counterexample and proof by reductio ad absurdum have not been assimilated by the mathematics teachers who took part in the experiment and deserve greater attention than they have hitherto received in the teaching of mathematics. We can see the same problem in the history of a mathematical idea related with the subject above explained.

REFERENCES

Balacheff N. 'Processus de preuve et situations de validation' Educational Studies in Mathematics, Vol.18 No.2, 1987.

Balacheff N. 'Une étude des processus de preuve en mathématique chez des éléves de Collége'.Thesis, Université Joseph Fourier I.N.P., Grenoble, France, 1988.

Brousseau G. 'Les obstacles épistémologiques et les problèmes en mathématiques'. Recherches en Didactique des Mathématiques, Vol.4, No.2, p. 165-198, 1983.

Fischbein E. 'Intuition, structure and heuristic methods in the teaching of mathematics'. Proceedings of the Second International Congress on Mathematical Education. Edited by A. Howson, Cambrige University Press, 1973.

Fischbein E. 'Intuition and Proof'. For the Learning of Mathematics 3, 2, p.9-24, November 1982.

Fourier J. 'Théorie analytique de la chaleur'. Gauthier Villars, Edition 1888.

Hitt F. 'Comportement de 'retourn en arrière' aprés la découverte d'une contradiction'. Thesis the troisième cycle, U.L.P. de Strasbourg, France, 1978.

Lakatos I. 'Proofs and Refutations (The Logic of Mathematical Discovery)'. Cambridge University Press, p.127-154. 1976

Robinson A. 'Non-Standar Analysis'. Amsterdam, North-Holland, 1966.

Vergnaud G. 'Cognitive and Developmental Psychology and Research in Mathematics Education: some theoretical and methodological issues'. For the Learning of Mathematics 3, 2, p.31-41, 1982.

A LOGO-BASED MICROWORLD FOR RATIO AND PROPORTION

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We report the results of the first part of a study to design, implement and evaluate a Logo-based microworld for ratio and proportion. The microworld provided pupils with pre-written Logo tools which could be used and explored in terms of their internal and external relationships; pupils were given opportunities to create and explore their own programs. The microworld was implemented in a class of 24 children with the researchers acting as teachers. Evaluation took the form of i. written pre and delayed post-tests; ii. audio recorded interviews (pre and post) with 9 pupils; iii. process data during the microworld implementation.

We begin with some attempt to identify the essential components of a computer-based microworld. In general, this will require identification of pupil initial conceptions so these can be worked with during the activity, carefully planned pedagogical intervention to 'impose' a mathematical perspective on the activity and some consideration of the range of contextual factors which are crucial to the learning process (see Hoyles and Noss, 1987). Also, such learning environments need to strike a delicate balance between exploration and structure; between allowing the child sufficient time and space to nudge up against the ideas embedded within the environment, and the attempt to maximise her chances of doing so. We have already experimented in well-defined and restricted mathematical domains with a small number of children; (see for example, Noss and Hoyles, 1988; Sutherland, 1987). The construction and investigation of children's interaction in such environments is problematic largely because any mathematical concept is part of an intricate network of concepts — so addressing one inevitably necessitates calling upon understandings of a whole range of other mathematical ideas. The ongoing work reported here describes in more detail than in our past studies the pupil perspective prior to engagement in the microworld and the pedagogical sequence as well as computer-based tasks. In addition it represents our first effort in microworld activity with a whole class rather than in an experimental situation with small groups.

We have chosen ratio and proportion as the conceptual domain. One reason for doing so is the considerable research effort which has been centred on this issue resulting in a comprehensive picture of the range of pupil responses to ratio questions to be expected in a non-computational environment (see for example Tournaire and Pulos 1985, Hart (1984). Finally our own research has indicated that, under the appropriate conditions, the computer makes a **qualitative** difference to what pupils can do (for example, in 'seeing the general in the particular') as well as influencing the strategies they choose to adopt and the skills they exhibit (Hoyles and Sutherland 1989, Hoyles and Noss 1989).

Our general aims were to utilise the power of the computer and the feedback it can provide, to provoke children:

— to use and get to know graphical 'objects' built according to proportional rules (which were initially not made explicit to the pupils)

— to engage in proportion-based situations and to construct figures in proportion on the computer.

— to come up against visual conflict if the 'rules' of proportionality are broken.

Our objectives were to:

— uncover and then build upon children's underlying intuitions and ideas about ratio and proportion rather than focussing on their ability to manipulate.

— design a sequence of computer-based activities in which children would first **use** proportional ideas with the hypothesis that such functional use (with appropriate structure) would lay the foundation for discrimination and generalisation given an environment which facilitates the linkages between intuitive actions, graphical outcomes and symbolic descriptions (Hoyles 1986).

— plan pedagogical interventions to promote these links, assist in computer-use and build in pupil discussion in order to provoke children to articulate their methods in natural language. Thus spoken and written natural language would act as a bridge between vague intuitions and the formal specifications needed to write a computer program or conversely be the means by which pupils could make sense of (discriminate) the meanings of the computer formalism.

— strike a balance between our own agenda and the children's own activities.

We thus planned that the computer would provide assistance with arithmetic operations; more importantly, it could offer cognitive scaffolding for making sense of the mathematical meanings of ratio and proportion. We particularly wanted to devise activities that would produce visual feedback to stand in conflict with common initial strategies (such as 'adding').

The microworld consisted of a set of activities Logo-based and paper and pencil, with some well-defined pedagogical agendas around which we focussed the children's activities. In thinking about design issues in computational environments, it is important to address a number of peripheral (to the specific intended mathematical learning) but important issues concerning the pupil/computer interface; such as familiarity with the computer, the creation and editing of Logo procedures, the syntax and meaning of variable, acquaintance with Logo's arithmetic operations and flow of control. We similarly recognise that turtle orientation is sometimes a source of confusion for children with limited Logo experience. We deliberately avoided having to address this explicitly in the activities whilst still working within the turtle graphics subset of Logo.

Methodology

The work which is reported here forms a component of a larger research study — the Microworlds Project[1]. The work reported here represents the results of the pilot study[2]. This took place in a single comprehensive school with a class of 24 children aged between 12 and 13 years. The class teacher had been a participant in the Microworlds project. All the teaching and organisation in the pilot study was however undertaken by the three researchers. The teaching experiment consisted of 6 sessions of 70 minutes duration in the school computer room (11 computers); children worked in pairs or exceptionally in groups of three. The children had very limited previous Logo experience (0-2 hours). The methodology to be adopted was:

— pre, post[3] and delayed post tests for a whole year group — the same test to be administered on each occasion.

— audio-recorded interviews of 9 of the 24 pupils in the experimental class in order to probe their answers: the choice of the nine was on the basis of their answers to the pre-test (to obtain a spread of apparent misconceptions), distribution of girls and boys (to represent the distribution within the year-group) and spread of attainment (as judged by the mathematics teacher).

— process data on microworld implementation consisting of: observational notes on the 9 pupils to assist the interpretation of the post-test results in terms of the activities undertaken during the microworld;marked homework assignments administered after each session;hard copies of the procedures written by all the pupils within the class.

The pre and post written tests were designed to probe children's intuitions and understandings about proportion from as wide a range of viewpoints as possible. We set out to investigate:

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[1] The Microworlds project is co-directed by the authors and funded by the Education and Social Research Council in UK. (1986-9). The aim of the research is to assess ways of using the computer to provoke mathematics teachers to reflect upon, and if necessary change, their practice. The project consists of the evaluation of an in service course for teachers in terms of attitude change and implementation together with microworld design.

[2] The main study will have a similar methodology but consist of two strands: teaching of the microworld (suitably modified as a result of the pilot) by the researchers to one class of 30 pupils aged between 13 and 14 years; teaching of the microworld to 3 classes: a class of 25 pupils aged 12 to 13; a class of 10 pupils aged 15 to 16; a class of 20 11-12 year olds (all of these classes are normally taught by teachers who attended the Microworlds course).

[3] No immediate post-test was given due to administrative problems.

— the influence of different contexts; that is to see how context might be used (or not) by the pupils in deciding upon their strategies and assessing the 'correctness' of their answers. Questions were set in three different contexts designed to be realistic for the pupils and thought to carry with them intuitions of enlarging or shrinking in proportion. The first question was a 'recognition' task involving the identification of a range of given rectangles which could be different sized plans of a swimming pool of given dimensions; the second question consisted of a set of paint-mixing problems; and the third a set of questions involving photographic enlargements (an example of this latter category is given in Figure 1).

— the effects of different mathematical structures; that is the distinction between scalar and functional relations (Vergnaud 1983), integral and fractional scale factors, and non-integral answers.

— the possible differential effects of the essentially graphical microworld on numerical problems (paint-mixing) and visual problems (photographic enlargements of rugs).

We also deliberately excluded questions where a simple 'doubling' strategy was appropriate. In the pilot tests, no calculators were available.

You collect photographs of rugs. You have just received a new set of photographs to add to your collection. You need to enlarge or shrink them to fit into the spaces in your catalogue. One of the 'new' lengths is unknown. Find the missing lengths (marked "?") on each diagram.

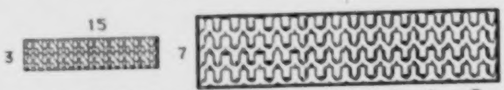
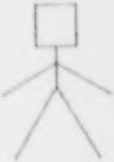


Figure 1: A Rug Task: Integral Functional - Multiplication: Question 3(b)

The Microworld

Two computational 'objects' formed the basis of exploration in the computer based activities. These tools were designed in order that by their use and examination the pupils could become aware of the important ideas behind ratio and proportion. One was a figure — LESLI — made up of variable parts as shown in Figure 2. The second was a fixed closed shape — HOUSE — using FD BK RT LT. The overall objective of the activities was for pupils to use and then discriminate the nature of the functional relations (i.e. multiplicative) within LESLI in order that 'families' of LESLI's would be in proportion; to recognise the nature of the scalar relationships needed (ie multiplicative) to achieve different sized proportional HOUSEs and to recognise — through visual feedback — the conflicting situations which would arise when non-multiplicative relations were employed in either context.



```
TO LESLI :SIZE
JUMP :SIZE
SHAPE1 :SIZE
LINE :SIZE* 0.4
SHAPE2 :SIZE* 1.5
LINE :SIZE* 0.6
SHAPE3 :SIZE* 2
END
```

Figure 2: Procedure and picture of LESLI

We provide below a resume of the six sessions. Each session was followed by written homework which provided researchers and pupils with ongoing feedback and formative evaluation.

Session 1: Pupils were given the constituent components of LESLI (SHAPE1, SHAPE2 etc.) and invited to use these to make patterns of their own choosing. Pattern generation was facilitated by the provision of procedures JUMP and STEP which respectively moved the turtle (without drawing) up and across the screen. The use of these procedures avoided the problematic issues of interfacing procedures and turtle orientation. The main pedagogical interventions for this session were to provoke pupils to use a wide a range of inputs (including negative and decimal); to give help on the technical issues of procedure definition and editing and to encourage collaboration and sharing.

Session 2: This session began with an introduction to the idea of using Logo's arithmetic operations to perform calculations. Pupils constructed LESLI as a whole by putting together the component parts. Our aim was to focus pupils' attention on the components of LESLI and the inter-relations between these components.The relationships within LESLI were all multiplicative so sets of LESLIs produced by using different inputs were **necessarily** in proportion. Pupils were then invited to explore with LESLI and create their own designs using different inputs. They were also asked to predict the size of the component parts of LESLI for specific values of the input.

Session 3: The aim was to encourage pupils to reflect on the way the procedure LESLI worked and to discriminate the necessary nature of the functional relationships between LESLI's constituent subprocedures in order to obtain sets of LESLIs all in proportion. First LESLI 83 was 'played out' away from the computer using the 'little people metaphor' as a class activity. Calculations were undertaken on the computer and leaving an 'open' answer encouraged leaving the calculation as 83 * 1.6. Pencil-and-paper tasks were

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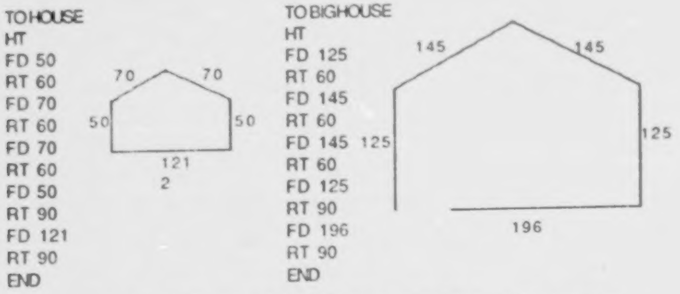
given in order to encourage reflection on the relationships between visual image,its formalisation (its procedure name) and its numerical 'size'. Pupils were asked to modify LESLI so that it had a large head, long arms etc. Our intention was to focus attention on the idea that **only** multiplicative modifications would produce classes of figures that were in proportion since strong visual images drew attention to the non-proportionality of figures constructed using different rules.

Session 4: A class discussion was held to consider how to make LESLI's arms shorter. Attention was focussed on how to label the lengths of the constituent parts of LESLI on a paper and pencil diagram and make explicit the multiplicative solution. Finally, a LESLI with a small head, PINHEAD, was given where the small head was generated by using a subtraction strategy; thus small input values would produce "upside down" heads, and focus pupils' attention on the non- proportionality of the resulting figures.

Session 5: Pupils were given a procedure for a closed shape — HOUSE — using FD BK RT LT . Pupils were asked to make bigger and smaller HOUSEs all in proportion. It was anticipated that attention would be drawn to the necessity of multiplicative scalar relationships since cognitive conflict would be generated on the adoption of non-multiplicative strategies (production of non-closed shapes or overlaps). Figure 3 illustrates computer feedback on the adoption of an additive strategy results and the obvious mismatch between the intended and actual outcomes.



```
TO HOUSE
HT
FD 50
RT 60
FD 70
RT 60
FD 70
RT 60
FD 50
RT 90
FD 121
RT 90
END
```

```
TO BIGHOUSE
HT
FD 125
RT 60
FD 145
RT 60
FD 145
RT 60
FD 125
RT 90
FD 196
RT 90
END
```

Figure 3: a)HOUSE ; b) BIGHOUSE adopting an additive strategy.

Session 6: We organised group tasks to work on differently sized HOUSEs of the pupils' choosing, in order to make the method explicit. The session started with a game in which pairs of pupils produced 'enormous' houses — in proportion to the original HOUSE — and challenged another pair to find how it had been generated given one length. The aim was that the pupils would: construct similar shapes where their scale factors was **not** obvious (in order to present a challenge for the opposite team); be forced by the rules of the game to negotiate their methods in pairs and make their methods explicit; devise methods to generate the rule of the 'opposite' pair given the original HOUSE and one length in the final HOUSE and finally compare their numerical results across pairs and defend their decisions. During the process of 'finding the rule' we encouraged pupils to work out the scaling factor iteratively using the computer; that is, in order to find the number to multiply 30 by to give 50, guess a number, try it and improve the guess.

Findings

The results of the pre and delayed post tests for the integral rug and paint questions are presented in Table 1[4]. Although these results are not spectacular the difference between the pre and the delayed post test results for questions 3b (see Fig. 1) and 3c is significant at the 5% level.

Question No.		Type of Question	Pre-Test % Correct (n=24)	Delayed Post-Test % Correct (n=25)	Ratio
PAINT	2a	INTEGRAL Functional (x)	16.7	34.8	3:9 = 2:?
	2b	INTEGRAL Scalar (x)	16.7	13.0	3:7 = 12:?
	2c	INTEGRAL Scalar (/)	8.3	17.4	10:15 = 2:?
	2d	INTEGRAL Functional /)	25.0	30.4	8:2 = 20:?
RUG	3a	INTEGRAL Scalar (x)	33.3	47.8	9:? = 4:12
	3b	INTEGRAL Functional (/)	0.0	30.4*	15:? = 3:7
	3c	INTEGRAL Scalar (/)	16.7	52.2*	18:? = 24:4
	3d	INTEGRAL Functional (/)	29.[illegible]	13.0	28:8 = 7:?

Table 1: Pre- and delayed post-test results for Integral Paint and Rug Questions (* denotes significant at th[illegible]% level)

More interesting than these overview statistics were the results of the interviews with the nine individual pupils. Of these nine pupils, six exhibited quite a major shift in the ways they attempted to approach the questions[5]. These shifts can be characterised as follows: a) from a perceptual to a more analytic strategy; b)towards a consistent strategy, and b) towards a multiplicative strategy. Analysis of these shifts will form the focus of our continuing work.

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[4] Almost all the pupils were unable to answer the non-integral rug and paint questions so this data is not presented below.

[5] Affective considerations: The remaining three pupils interviewed could all characterised by an attitude which appeared to be unconcerned by whether their solutions were correct. They still persisted in using mathematically inconsistent strategies. It was noticeable that all three were not involved at any deep level in the microworld activities.

Conclusions

Our data has led us to propose a number of substantial changes in the design of the microworld — in terms of sequence but also in terms of making our definitions and methods more explicit. We also now see that in order to exploit the visual feedback and understanding conflicting evidence during computer activity, an understanding of the mathematical meaning of proportion and a recognition of the need for consistency is necessary. We also intend to: stress the equivalence of comparing similar figures in a scalar and in functional ways and that both ways give the same 'result'; build on (rather than avoid) intuitive doubling and deliberately try to forge the link between doubling and 'times by 2'; specify the use of calculators in the pre , post and delayed tests, for all questions involving non -integral scale factors and specify that a calculator should **not** be used for integral questions (rather than leaving the decision open).

Our tentative conclusions from the pilot study are that our microworld achieved some limited success. We are reasonably confident that, given the modifications above, the main study will be able to generate an appreciation of the meaning of proportion and its formalisation in terms of multiplicative operations. In the work undertaken already for the main study we anticipate further interesting shifts — for example, from random number pattern spotting to searching for the ratio pattern, from senseless answers to appreciating the context of the questions. Results of the main study will be presented at the conference.

References

HART, K.M., (1984), *Ratio : Children's Strategies and Errors,* Windsor: NFER-Nelson.
HOYLES, C., & NOSS, R., (1987), Children working in a Structured Logo Environment: From Doing to Understanding, *Recherches en Didactique de Mathematique,* Vol.8.,1.2, p.131-174.
HOYLES, C., & NOSS, R., (1987), Synthesising Mathematical Conceptions and their formalisation through the Construction of a Logo-based School Maths Curriculum, *International Journal of Mathematical Education in Science and Technology,* 18, 4, July/August, p.581-595.
HOYLES, C., & SUTHERLAND, R., (1989), *Logo Mathematics in the Classroom,* Routledge.
HOYLES, C., (1986), Scaling a Mountain - a study of the use, discrimination and generalisation of some mathematical concepts in a Logo environment, *European Journal of Psychology of Education,* I, 2, p. 111-126.
NOSS, R., & HOYLES, C., (1988), The Computer as a Mediating Influence in the Development of Pupils' Conception of Variable, *European Journal of Psychology of Education,* III,3, p.271-286.
SUTHERLAND, R., (1987), What are the Links between Variable in Logo and Variable in Algebra?, *Recherches en Didactiques de Mathematiques,* 8.
TOURNAIRE, F., & PULOS, S., (1985), Proportional Reasoning: A Review of the Literature, *Educational Studies in Mathematics,* 16, 181-204.
VERGNAUD, G., (1983), Multiplicative Structures, in LESH, R., & LANDAU, M., (Eds.), *Acquisition of Mathematics Concepts and Processes,* Academic Press, NY.

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THE FACILITATING ROLE OF TABLE FORMS IN SOLVING ALGEBRA SPEED PROBLEMS: REAL OR IMAGINARY?

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The instruction of how to solve mathematics problems uses several auxiliaries, including the table form. We tested the effects of using table forms on the solution of speed problems. The results refute the common belief that table forms facilitate problem solution: Half of all the solutions that were based on a table form were incorrect, and when students used table forms in 1, 2, or 3 problems, only 9.9%, 12.5%, and 17.1% of them were successful, respectively. The examination of the table forms revealed that most of them were faulty. The results of no facilitating effects is attributed to the inherent drawback of table forms, which comprise Nondirect Relations but neither hint nor provide for their prerequisite inference by the INFER schemas.

INTRODUCTION

The analysis of algebra speed problems (Harel and Hoz, forthcoming) identified three kinds of relation that may be found in problems dealing with rectilinear) motion: Basic, Direct, and Nondirect. The **Basic** relations indicate whether elementary temporal and spatial attributes (such as starting times, terminal points, and direction of motion) of the moving objects are same or different. To solve a speed problem its representation must incorporate th[illegible]ions it includes. To achieve this it is not enough though to use basic relations, and other relations have to be represented as well. Of these, some can be encoded into the representation directly, while those which are implicit in the problem statement have to be first inferred from the basic relations. This inference is a prerequisite for the solution of many a type of speed problems and it may be relatively complex and difficult to achieve. The relations other than the basic ones are classified as either direct or nondirect. **Direct** relations can be derived directly from the problem statement and represented without using basic relations. **Nondirect** relations cannot be derived this way but rather inferred from basic relations to be represented. All three types of relation pertain to the concepts distance and time, but only the Direct pertains for the concept of speed. For example, "Car 1 was on its way 3 hours more than car 2" is a direct relation, since it can be encoded directly with no need to infer by how much one duration is larger than the other. "Car 1 started from city A 5 hours before car 2, which arrived to city B 3 hours after car 1" is a nondirect relation, since the same relation ("car 1 was on its way 3 hours more than car 2") has to be inferred from the stated basic relations.

Cognitive analysis illustrated the importance of structure variables in problem solving (e.g., Goldin, 1984) and ours (Harel and Hoz, forthcoming) has identified the **INFER DURATIONS RELATION** and **INFER DISTANCES RELATION** schemas as the mental inference mechanism for nondirect relations. The latter are similar to the Part-Whole schema (Riley, Greeno and Heller's, 1983, and Nesher, Greeno and Riley's 1982), and can accounted for the difficulty of speed problems that students have at all educational levels, in both formal school algebra and nonmetric tasks (e.g., Siegler and Richards, 1979; Wilkening, 1982; Mayer, Larkin, and Kadane, 1984; Reed, 1984; Gorodetsky, Hoz, and Vinner, 1986; Goldenberg, 1989).

The observed difficulties that many students have in solving mathematical problems inspired mathematics educators and psychologists to propose auxiliary means and models. Most of these concentrate on parsing the problem statement into its components that correspond to the problem's elements and translating each into an equation. From both the theoretical and practical aspects Polya is the most salient proponent of such means. He proposed (Polya, 1957) the use of a table form and highly recommended to use it to present the relations between the values of one variable (stated as the "givens"), and to facilitate obtaining the relation between the values of a second variable (stated as the "condition"). The latter, when expressed as an equation models the problem (an example is presented in the Discussion part).

Table forms are very popular and used by many mathematics teachers, who believe them to be helpful in deriving the equation(s). This may be especially true for the weak students who cannot understand let alone solve most algebra problems, but seem capable of coming out with some equation as a result of using a table form. There are though teachers who reject the use of table forms. They view the teacher's primary role as developing student thinking and problem solving ability, and do not believe in an easy life in mathematics classes. They want their students "to think hard" and discover the solution on their own, and they contend that table filling involves none of these, being a technical and automatic way of problem solving of bypassing the important functions of "understanding" or "thinking".

Within the framework of the research on the solution of speed problems (Goldenberg, 1989; Harel and Hoz, forthcoming; Hoz and Harel, 1988) we addressed the question whether table forms can facilitate the solution of speed problems. It emerged when table forms were examined in light of the central role that the INFER schemas play in the solution of problems. It became evident that Polya's proposal does not take cognizance of the INFER schemas, and his treatment lacks in three respects which may render the application of table form useless. (a) The columns of a table form do not represent the Basic but only the Direct and Nondirect relations. (b) He had not distinguished the recognition of basic relations (that he considered one of the "givens") from the inference of the nondirect relations from them. (c) He never mentioned nor elaborated on how this inference is to be made. The hypothesis tested was that this type of auxiliary is not helpful in the solution of speed problems as it was designed and is claimed to be.

METHOD

The subjects were 178 students enrolled in three 9th, seven 10th, and three 11th grades in a comprehensive high school in Beer-Sheva. The tests were administered to whole classes, and the instructions required only to set the equation(s) but not to solve them. The students were neither told nor hinted as to what auxiliaries to use, but were required to provide full explanations to their answers. Enough time (up to one class period) allowed all the students to complete the test.

To test the research hypothesis we used five variables. The independent variable was the nature of Time Basic Relations from which Time Nondirect Relations had to be inferred (explained later). Three variables were measured for each solution: (i) whether it included a table form, (ii) whether it had congruent table and equation(s), and (iii) whether it was correct (i.e., had a correct equation(s)). Three dependent variables were derived, which record the number of solutions in each test (0, 1, 2, or 3) with each of the features: (A) including table forms, (B) having congruent table and equation, and (C) having a correct equations. Ordered in this way, each of them characterizes a more progressive phase in the solution if the test had at least one table form.

The test problem involves two cars, each going at a different speed from one place to another: "Two cars go from city A to city B, *a* km apart. Car 1 leaves city A *b* hours after car 2. Car 1 arrives at city B *d* hours before car 2. Car 1 is *c* km/h faster than car 2. Find the speed of each car." This is the simplest problem type possible for two cars, in which (1) the Distance Nondirect Relation can be easily inferred, (2) the same Distance Speed Relation is that one car is faster than the other, (3) only one Time Nondirect Relation has to be inferred from two pairs of Time Basic Relations that pertain to the starting and arriving times. The city names and the values of *a, b, c,* and *d* were specified, and the latter as well as the phrases *before* and *after* were different in each problem, to provide for the experimental design. The three possible time basic relations are: the cars started (arrived) together, car 1 started (arrived) first, and car 2 started (arrived) first. These yield 9 different combinations, three of which were represented in each of the four test forms that were distributed equally among the students in each class.

RESULTS

To test the hypothesis the tests were classified into the following two-way table. In theory the three dependent variables are unrelated to each other, but in practice they were, as the table shows: the more advanced phase in the solution a feature represents, the smaller the number of solutions that have it.

FEATURE OF SOLUTION	NUMBER OF SOLUTIONS IN TEST				TOTAL NUMBER OF	
	0	1	2	3	TESTS	SOLUTIONS
A Has table form	11	8	0	70	89	218
B Has congruent table and equation	15	13	22	39	74	174
C Has correct equation	15	20	27	12	59	110

These figures show that when a table form was used, the chance is .83 (174/210) that equation and table form are congruent, and the chance is .63 (110/174) that the equation is correct. Hence, the overall chance to obtain a solution is about .50.

The first observation in regard to students is that only half of them (89 out of 178) attempted to construct table forms (and 11 of these did not manage to write any equation). The majority (89.7%) of those who constructed table forms, did so consistently for all three problems (70/89). Of these, 17.1% (12/70) obtained three correct equations, 37.1 % (26 out of 70) obtained two correct equations, and 27.1% (19/70) obtained one correct solution. The rest (18.7%) did not get any correct equation. Therefore, despite the success of most students in basing their equations on the tables, the large percentage of incorrect equations reflects the general failure to construct table forms that reflect the Time Nondirect Relation.

Of the students who constructed table forms for all problems, only 9.5% could derive 3 correct equations. Of those that had three congruent tables and equations, only 30.8% (12/39) wrote correct equations. Further analysis revealed a positive linear relation between the number of table forms in the test and the number of correct solutions (the more table forms constructed the higher the success rate): When 1, 2, and 3 table forms were constructed, 9.9%, 12.5%, and 17.1% of the solutions were correct, respectively. Also, in 83.1% (74/89) of the tests at least one equation was congruent with the table, and in 79.7% (59/74) of these at least one equation was correct. Therefore, of all the students who attempted table forms (typically in 3 solutions), 66.3% got a correct solution to at least one solution.

DISCUSSION

The results of this research clearly support the hypothesis that table forms do not facilitate the solution of the simplest type of speed problems. Using table forms, the chance of a student to obtain at least one correct solution for three identical problems depends on the number of table forms constructed, and lies between .10 and .17. The chance of a solution to be correct when it is based on a table form are about .50. These estimates can at best be the upper limit for these probabilities, since only half of the students used table forms. Extrapolating from their results to those who (for unknown reasons) preferred not to use table forms, it seems plausible to estimate that 3/4 of the latter would not construct appropriate table forms.

The reason why table forms, despite being consistently used by half of the students, were found unhelpful lies in the nature of the INFER DURATION schema and its role and function in the solution. We argue that if table forms were designed to aid students glean the meaning of the problem by arranging the relations in an orderly (and expectedly) helpful manner, then it is not the case that our students used table forms ineffectively. Students failed to solve the problems because the relations were not appropriately represented in the table form. Those students could not be helped by table forms, that neither hint at nor provide for the instantiation of the INFER schema for the inference of the Time Nondirect Relation (or Distance Nondirect Relation in other problems). The following table form for our test problems illustrates this argument.

		VARIABLE	
	distance	speed	time
car 1	a	a/x	x
car 2	a	$a/(x+b+d)$	$x+b+d$
EQUATION		$a/x=a/(x+b+d)+c$	

This table form includes the Time Nondirect Relation but not any Time Basic Relation. The values in the speed column prove that the inference of the former relation is a prerequisite, that cannot be bypassed, for the derivation of the Speed Direct Relation. Table forms are useless for students who lack the INFER DURATION schema in their knowledge base. This example clearly indicates that the construction of table forms is not a matter of automatically filling in the variable values, nor is it a warranty for correct problem solution (even to isomorphic problems, one of which was successfully solved).

The conclusion to be drawn from the results of this research is that if auxiliaries are to be of any help in problem solving they must be based on theoretical cognitive analysis of the solution processes (and therefore may be domain-specific) in the first place. They also have to address specific factors and processes that were identified by cognitive analysis or empirical findings as needing help.

REFERENCES

Goldin, G.A. (1984). Structure variables in problem solving. In G.A. Goldin (ed.) Task variables in mathematical problem solving. Phil[illegible]ia, PA.: Franklin Institute Press.

Goldenberg, E. (1989). The development in the ages 13-15 of the three-concept relationship between speed, time and distance. Unpublished Masters thesis. Beer-Sheva: Ben-Gurion University of the Negev.

Gorodetsky, M., Hoz, R., and Vinner, S. (1986). Hierarchical solution models at speed problems. Science Education, 70, 565-582.

Greeno, G.J., Riley, M.S., and Heller, J. (1983). Development of children's problem solving ability in arithmetic. In H.P. Ginsburg (ed.) The development of mathematical thinking. New York: Academic Press.

Harel, G., and Hoz, R. (forthcoming). Structural analysis and isomorphism of speed problems. TheJournal of Structural Learning.

Mayer, R.E., Larkin, J.H., and Kadane, J.B. (1984). A cognitive analysis of mathematical problem solving ability. In R.J. Sternberg (ed.) *Advances in the psychology of intelligence*. Hillsdale, N.J.: Erlbaum.

Nesher, P., Greeno, J.G., and Riley, M.S. (1982). The development of semantic categories for addition and subtraction. *Educational Studies in Mathematics, 13,* 373-394.

Reed, S.K. (1984). Estimating answers to algebra word problems. *Journal of Experimental Psychology: Learning, Memory and Cognition, 10(4),* 778-790.

Polya, G. (1957) *How to solve it*. New York: Basic Books.

Riley, M.S., Greeno, J.G., and Heller, J.I. (1983). Development of children's problem solving ability in arithmetic. In H.P. Ginsburg (ed.) *The development of mathematical thinking*. New York: Academic Press.

Siegler, R., and Richards, D.D. (1979). Development of time, speed, and distance concepts. *Development Psychology, 15,* 288-296.

Wilkening, F. (1982). Children's knowledge about time, distance, and velocity. In W.J. Friedman (ed.) *The developmental psychology of time*. New York: Academic Press.

THE LEARNING OF PLANE ISOMETRIES FROM THE VIEWPOINT OF THE VAN HIELE MODEL

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Abstract

The aim of this communication is to suggest a new application of the Van Hiele model of reasoning. We present a description of the Van Hiele levels for the learning of plane isometries, and some examples of activities for each level. We have obtained this theoretical description of the Van Hiele levels from experiments in school settings carried out with primary school students and with pre-service primary school teacher.

Introduction

The Van Hiele model of reasoning claims that there exist several levels of reasoning for the students; one of its main claims is that, for successful teaching, it is necessary to take into account the students' current level. Therefore, one of the main aims of the Van Hiele model, is to analyze each area of geometry (or of mathematics in general) and to characterize each level of reasoning using elements belonging to a given area, in order to develop teaching units for the classroom. In the existing literature there are general descriptors (see Usiskin (1982), Burger & Shaughnessy (1986), Hoffer (1983), and Fuys, Geddes & Tischler (1985)) and also specific descriptors and teaching units focused on several areas of plane geometry, such as polygons, angles or surfaces (see Fuys, Geddes & Tischler (1985) and Scally (1987)). But there are other important topics which have not yet been investigated; one such topic is geometric transformations and, in particular, plane isometries; although Hoffer (1983) and Alsina, Burgues & Fortuny (1987) do present descriptions of the levels in terms of plane isometries, they are simply theoretical statements lacking any further practical application. In our current research we have continued the study of the Van Hiele model that we began some time ago, with relation to measurement and spatial geometry (see Gutierrez & Jaime (1987a) and Gutierrez, Fortuny & Jaime (1988)) by working on plane isometries.

The results that we show here have been obtained from our work over several years on teaching plane isometries to primary school pupils and to future teachers in Valencia (see Gutierrez & Jaime (1987b and 1988)). Our wish to provide pupils with activities according to their reasoning abilities has led us to use the Van Hiele model. Therefore, we have first determined the characteristics of translations, rotations and symmetries for each Van Hiele level and, within each level, those corresponding to the learning phases which allow access to the higher level. Secondly, we have designed teaching units for each isometry, taking into account these characteristics.

Now we shall present the general characteristics of each level, related to plane isometries. As we think that it will be clearer if we give examples of just one isometry instead of using all three symmetries for different examples, we will confine ourselves to the translations in the examples.

Of the various opinions on the number of levels of the Van Hiele model, we assume (see Gutierrez & Jaime (1987a)) the existence of four levels of reasoning, namely, (1) recognition, (2) analysis, (3) classification and (4) deduction. Table 1 shows a summary of the Van Hiele levels of reasoning for plane isometries.

Now we shall make a detailed description of the characteristics of the four levels and the most significant results of our experiments. The activities on plane isometries that we propose to the students include, in general, the use of cut-outs, to promote active learning and to avoid difficulties caused by the children's lack of drawing ability.

Table 1 The Van Hiele reasoning levels in plane isometries

Level 1	Visual identification of translations, rotations and symmetries Static recognition identification of isometric figures Dynamic recognition The movements are carried out automatically
Level 2	Experimental discovery of the elements and basic properties of the isometries The isometries are made and identified by means of their elements and basic properties
Level 3	Experimental deduction of relations and properties of the isometries Justification of properties and relations already known Formal definition of translation, rotation and symmetry Products and decompositions of isometries are determined
Level 4	Global insight of plane isometries Properties are proved formally, the structure of group is taken into account, the relations existing between the isometries are generalized.

LEVEL 1 There are two ways to beginning to discover the plane isometries static and dynamic The static approach consists of the visual recognition of figures which correspond to each other under an isometry, this recognition includes the use of figures arranged in non-standard positions In the dynamic approach, the students move the figures physically, in the early phases of this level they use some devices (rulers, discs, mirrors, computers, folding,), and in the later phases the students can begin to perform the movements without those tools, by remembering what they have done before

Some *types* of activities for translations on level 1 are

- Giving examples and non-examples of translations
- Moving figures or objets along a ruler or a straight line
- Asking pupils to talk about the differences between translated and non-translated figures, to do so, they can use a ruler and make the movements physically or tell by looking at the figures

- Asking the students for some examples of translations from his environment
- Translating a figure so that one of its segments maps onto another given segment

It is evident that when students use visual recognition (a behaviour characteristic of the first level), they use the elements of isometries (directed segment, center, reflection line) and some of their basic properties, but they will only become conscious of them when they have reached the level 2

LEVEL 2 The work with the students at this level begins with the discovery of the basic elements and characteristics of each isometry Directed segment and parallelism (translations), center, directed angle and movement along circumferences (rotations), reflection line, equidistance, perpendicularity and inversion (symmetries) When identifying which figures correspond to each other under an isometry, at this level the students do not base their reasoning only on visual recognition, but they also verify the presence of the basic properties of the identified isometry, this allows the students to use ruler, compass and protractor to move points of the figures

However, the students do not relate the properties to each other, that is, they have not yet built up the network of relations; consequently, they are not able to determine minimal sets of properties that characterize an isometry and, theren, they cannot properly define isometries

One typical piece of behaviour observed in the early phases of level 2 is to expect different images after moving a figure under the same translation when the origin of the arrow has been placed on different points of the figure (see figure 1)

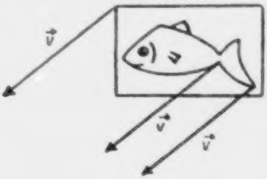

Figure 1

After experimentation they realize that the result will be the same, but they do not understand why

The absence of the network of relations can also be seen in the way students manipulate two figures to check if they correspond under a specific isometry Moreover, the students do not realize that they can locate the whole figure image under a given isometry when they know the image of two points of the figure (see figure 2)

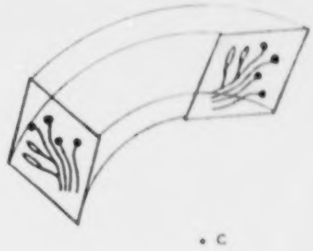

Figure 2

At level 2, the students learn to distinguish and to use the characteristics of translations (length, slope and direction); they also discover by experimentation other basic properties, such as parallelism between the corresponding figures When working with squared paper, students can also discover the coordinates of the arrow defining a translation, and they can describe them by means of whole numbers qualified by words such as right/left up/down (if students already know the negative numbers, they can use them) They can find products of translations and deduce from experimentation some properties, such as commutativity

With respect to rotations, some of the facts that the students will discover at level 2 are equidistance from the center, variation of slope (according to the rotation angle) of the rotated figure, the importance of angle direction and the existence of equivalent rotations The students can also handle products of rotations with the same center and discover some algebraic properties

As for symmetries, the students will discover equidistance and perpendicularity with respect to the reflection line of two symmetric points They also recognize other properties such as the parallelism between the segments that join several points and their respective images under a symmetry, the inversion of figures, the fact that the position of the image of a line varies according to its position relative to the reflection line, etc

There are some examples of different *types* of level 2 activities for translations

- Performing a translation given its directed segment
- Performing a translation on squared paper, given the coordinates of its directed segment (for instance, 3 squares to the right, 5 squares down)
- Completing several frieze patterns from the same figure, by means of translations whose arrows differ only in slope, length or direction Comparing the results and discussing the differences
- Checking whether two figures correspond to each other after a translation and, if they do, finding its arrow
- Obtaining products of translations and observing the results

LEVEL 3 At this level the students have already acquired the ability to relate the properties they already know and to discover new properties by experimentation and informal deductive reasoning They give definitions for each isometry, that is, they identify minimal sets of sufficient conditions to characterize an isometry They can give informal proofs for properties discovered at level 2

The students now know the minimal number of point-images of a figure needed to locate the whole image, and can justify this

With respect to the product of isometries, students can deduce the result of products of two symmetries or two rotations This will allow them to begin to build up a network of relations between various isometries in the later phases of level 3 (because they can find products which include different kinds of isometries) and to acquire a global understanding of isometries when they reach level 4 It is also possible at this level to work with glide reflections and to discover or deduce some of their properties from the knowledge they already have of translations and symmetries

In the later phases of level 3 the students can handle the general decomposition of isometries and, in several cases, obtain all the possible solutions (infinite sometimes)

As for rotations, the students can deduce that the perpendicular bisector of a segment is the set of the centers of all the possible rotations which map one endpoint of the segment onto the other; in this way, they will be able to discover the centers of rotation and will understand the meaning of the usual algorithm to discover the center of a circle

There are some *types* of level 3 activities about translations

- Finding products of several translations and discovering, from the coordinates of their directed segments, the coordinates of the resulting directed segment Generalizing and justifying the result
- Decomposing a translation into several products of translations (and justifying that there are infinite possibilities)
- Decomposing a translation into two symmetries, a) when one reflection line has been fixed, b) when no reflection line has been fixed Discovering and comparing the number of possible solutions in each case
- Predicting and justifying the result of the product of a translation and a rotation

LEVEL 4 The main activity which students develop at this level is formal and consists in deducing and proving complex properties and theorems which in the previous levels were out of the students' reach

These are some of the facts which must be used in the activities belonging to level 4, because they help to acquire a global insight of isometries

- The group structure of the plane isometries as a basic tool

- The Classification Theorem of the Plane Isometries (every isometry is equivalent to a product of at most three symmetries)
- Equivalent movements, decompositions and products
- Given the characteristics of several isometries, identify the movement which results from their product

References

Alsina, C., Burgues, C., Fortuny, J.M. (1987) *Invitación a la didactica de la geometria* (Sintesis Madrid)

Burger, W.F., Shaughnessy, J.M. (1986) Characterizing the van Hiele levels of development in geometry, *Journal for Research in Mathematics Education* vol 17, pp 31-48

Fuys, D.; Geddes, D., Tischler, R. (1985): *An investigation of the van Hiele model of thinking in geometry among adolescents* (final report). (Brooklyn College: N. York)

Gutiérrez, A.; Fortuny, J.M.; Jaime, A. (1988) Van Hiele levels and visualization in three dimensions, paper presented at the *"Visualization Topic Group"* (6th I.C.M.E., Budapest), preprint

Gutiérrez, A.; Jaime, A. (1987a) Estudio de las caracteristicas de los niveles de van Hiele, en Bergeron, Herscovics & Kieran (1987) *Proceedings of the 11th International Conference of the P.M.E.* vol 3, pp 131-137

Gutiérrez, A.; Jaime, A. (1987b) Estudio sobre la adquisición del concepto de simetria, *Enseñanza de las Ciencias,* n° extra, pp 365-366

Gutiérrez, A.; Jaime, A. (1988) Errors when making symmetries, en Goupille et al (1988) *Compte Rendu de la 39ᵉ Rencontre Internationale de la C.I.E.A.E.M.* (Edit. de l'Univ de Sherbrooke Canada), pp 400-404

Hoffer, A. (1983) Van Hiele based research, en Lesh & Landau (1983) *Acquisition of mathematics concepts and processes* (Academic Press N York), pp 205-227

Scally, S.P. (1987) The effects of learning Logo on ninth grade students' understanding of geometric relations, en Bergeron, Herscovics & Kieran (1987) *Proceedings of the 11th International Conference of the P.M.E.* vol 2, pp 46-52

Usiskin, Z. (1982) *Van Hiele levels and achievement in secondary school geometry* (ERIC USA)

Representation and Contextualization

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Summary. This paper presents a theoritical framework on how engineers and technicians make an efficient use of mathematics. Inspired from several cases reported on arithmetics, it introduces the notion of contextualised mathematics. Application and contextualization are contrasted from examples. The role of representations will be examined in this conjunction. This paper is basically theoritical in the sense that it determines new research avenues on the fondamental issue of professional training in mathematics and try to define some of the major factors concerned.

Introduction: a remark on the notion of contextualization.

The notion of representation will be understood as it is defined in Janvier (1987) [see the papers of Kaput, Mason and Goldin]. As for the notion of contextualization, the aim of this paper is precisely to make it explicit and to relate it to the notion of representation.

The **context** is often interpreted as the set of conditions or of propositions that at some point in time "organize" the meaning of a concept. According to such a definition of the notion of context, the development of a concept (and its learning) is necessarily associated with a double process involving de-contextualization and re-contextualization in which a notion gains meaning through a series of more and more refined settings. I do not deny the validity of such an approach but the issue at stake in this paper leads us to **depart** from this interpretation.

In fact, the notion of context to which I refer in this paper, brings us outside of mathematics. **Context** will be synonymous to **situation**. It will be regarded as the "concrete" **support** from which a mathematical concepts is derived. It basically presupposes that many basic mathematical ideas are abstracted form the real world. The process involves mental images that are close to the reality, close to the observed objects or relations.

The aim of the paper.

This paper will present the rationale behind a research that I have just strarted on how engineers and technicians make use of mathematics. Inspired from several cases reported on arithmetics, I will try to make explicit what could be meant by contextualised mathematics. This will lead me to question the notion of application. Finally, some experimental details will be provided. This paper is basically theoritical in the sense that it determines new research avenues on the fondamental issue of professional training in mathematics and try to define some of the major factors concerned.

The case of arithmetic.

Terezinha N. Carraher, Analucia D. Schliemann, Jean Lave and others (see references) have studied arithmetic "users" such as supermarket shoppers, young market merchants in developing countries, illiterate carpenters, bookkeepers, warehouse workers, lottery ticket sellers.. is order to discover to what extent arithmetic is used, the way it is used and how successfully?

These research projects has adopted a combination of observation techniques and interview procedures that do not rely on the users' impressions or beliefs. Their conclusions converge. As Lave, Murtaugh & de la Rocha (1984) puts it: "*There are evidence that workers make calculations which are arithmetically more advanced than they had the opportunity to learn at school.*"

These calculations are made almost error-free. However, what appears to me as more important, is that the algorithms and the procedures used are different from the ones introduced in school. First of all, the exists a tacit rule:"no pencil, no paper". Mental arithmetic is predominant. This explains that computation methods turn out to be "primitive". But, since being dismissed can be the consequence of a mistake, there exists a form of "priority" for using methods which will ensure success and with which one will feel confident.

Let us examine what is meant by primitive. For instance, multiplication is performed as a repeated addition. Three coconuts at 80¢ each will cost: 80, 160, 240: $2.40. For calculating how much are 12 melons at 50 cruzeiros, the youngster more or less mentally takes them "two by two" and counts: 100, 200, 300, 400, 500, 600. Sometimes, the procedure is more tricky: 10 times 35 will be carried out in the head as: (3 +3 +3) times 35 plus 1 time 35 which is equivalent to 105 (3 times 35) + 105 +105 (315) +35... 350. The technique of adding a "0" seems to be suspicious. As it has just been pointed out "**primitive**" does not necessarily mean **simple**. It may turn out to be associated with **a certain degree of complexity**.

In ratio and proportion problems, going back to the value of a unit is quite rare. Let us give an example. If 3 melons costs $6.00, how much are 9. In that case, the value of 1 melon ($2.00) will not be evaluated. "Users" will consider that they have three times more melons and that consequently they will pay three times more: $9.00. Some combinations of numbers will make such a procedure often difficult but it is by far prefered to the other one that consists in finding the rate, in other words, the price for a unit or the unity. This was already pointed out by Freudenthal(1983), Vergnaud (1979) and others.

This research orientation seems very inspiring because a form of arithmetic seems to be **constructed on the basis of particular needs**. There seems to exist a school arithmetics and a contextualized one It is possible to analyse further the differences between both but I will restrict myself to only one element: **the problem solving dimension of the tasks must not be overlooked.** In fact, we cannot imagine contextualization without problem solving whether the problem is solved on the spot or a known technique is being re-utilized with adequate adjustements. The problem solving "environment" seems to induce the"**informal ways**" of doing mathematical calculations *"which have little to do with the procedures taught in school...and that are more effective"*[Carraher et al.(1985)].

The notion of contextualization.

However correct the explanations we find in the litterature may be, it seems to me that the fact that the **computations are contextualized** is not sufficiently emphasized In other words, the descriptions of the users' performance do not take enough into account the inter-actions between the context and the calculations. In fact, calculations are not made with abstract numbers but rather on quantities, magnitudes or measures. More precisely, numbers are processed in the operations without loosing their situational connotations. This equally means that the context plays an active role. A multiplication problem becomes an addition problem as the melons (either actual as a mental images) are used as support for the reasoning. The question: *"How much for one?"* is avoided since combining the units (to obtain a rate) cannot be supported by a contextual entity which would be derived from the combination of observed numerical entities.

Contextualized mathematics

If the-man-in-the-street uses arithmetic his own way or contextually, is it possible that scientists, engineers or technicians do the same with mathematics in solving efficiently their problems whether when they solve equations or when they make use of functions, integrals, derivatives...? **This is precisely the question the current research will address.**

Trying to ask this question brought me back "in spirit" to my university years. Indeed, I have remembered how much I was frustrated by a Professor teaching electronics and who could amazingly juggle with trigonometry and complex numbers. I had the conviction that he was appplying mathematics when carrying out his circuits analyses. However, either this was not my mathematics or we were not applying it the same way. That is probably these memories that led me to check with electrical circuits and people around me.working on them.

Solution of an electricity circuit.

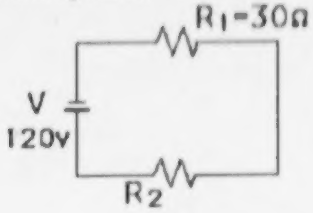

Figure 1

In the sort of easy problem presented in figure 1 and that is found at the end of all chapters on elementary electricity, the students are asked to find the resistance of resistor R_3, given that the current is 3 amps and the battery voltage 120 volts. The internal resistance of the battery is negligible.

In the traditional way to solve such a problem., the students are expected to set up the equations. To start with, it needs little inspiration to write down

$$V = V_1 + V_2.$$

And then, they can write: $V= R \bullet i$ and $V= R_1 \bullet i + R_2 \bullet i$. The last equation can be derived in many ways. The main goal to reach is the substitution of values in it to get:

$$120 = 3 \bullet 10 + 3R_2$$

And solving:

$$120 - 30 = 3R_2$$
$$R_2 = 90 / 3 = 30$$

The point to **emphasize** here is that a lot of equation handlings are performed **without resorting** to the circuit diagram. It would be very interesting to know more precisely what is the real contribution of this diagram in solving the problem. But, at any rate, the vast majority of physics teachers would not write down equations when it comes to solve such a simple electricity exercise. The equations would be replaced by "**the diagram**" itself which will be used to combine the quantitative relations between the variables involved. The **context** here can be considered as a **mixture of the well known basic current laws and the diagram** which guides and supports the reasoning. The **equations implicitly used by them (and having a strong schematic content) will differ** from those of the students that are mostly "mathematically inspired" in the sense that they are totally dependent on the allowed algebraic transformations: to put the unknown on one side...

In constrast, when our laboratory technician inspected the circuit, a total resistance came to his mind going with a voltage drop in two steps. Indeed, 3 amps for a voltage drop of 120 volts "requires" 40 Ω. The split 30 Ω and 10 Ω (which gives 40 Ω) appears quite clearly on the diagram. In short, the fundamental relations that are conveyed by the equation are readily visible on the diagram. In this sense, the diagram replaced the equation and was most likely used as the mental images or the concrete objects which help the young market merchants.

I am inclined to believe that each solution on the diagram (or without equation) should be examined in order to discover how the fundamental relations between voltage, resistance and current are articulated in conjonction with the diagram. For instance, you should scrutinize the way you or your friends would solve the above circuit. While in the arithmetic examples, standard computations that would bring the users too far from the context were avoided for reasons of certainty, here the equations are dispensed with because they appear extraneous to the reasoning. Can equations in such cases always be avoided? With more complex circuits, I imagine that some equations would be partially written down and a fair amount of work towards the solution would be carried at the diagram level. However, the last statement is only a hypothesis suggested by the skillful reasoning of my electronic professor. From the few readings I made on the topic (reasoning in electricity) the issue as to how the diagram comes into play is overlooked: an interesting research orientation. Let us clarify further the notion of contextualisation.

Contextualization

Applying mathematics is generally **associated with setting up equations or formulae and solving them**. At a more basic level, only a simple arithmetic operation or a proportionality relation will be set up. Nevertheless, and this is what I consider as being most important, there exists an epistemological tradition implicitly accepted by the scientific community which assigns to mathematics a precise role that it plays with respect to science in general. According to this conception, mathematics is **at first learnt** in the mathematics lessons and then **applied** in the science lessons. In fact, mathematics is as every science a generalizable knowledge. Always according to this conception, the domain in which mathematics is meaningful and in which it can be used do not change the basic mental operation performed when problems are solved. It is moreover considered that mathematics points towards the genuine solution, the others being regarded as partial or inadequate. This has brought about a very well defined scenario: applications take place in the solutions of the end-of-chapter problems.They conduce to the "writing down" of a few equations which are then solved more or less successfully. Too often, the modelisation by which the equations are at first associated to a phenomenon belong to the teacher exposition. The initial model is reduced to the one mathematics can treat. The underlying analysis through which contextual elements are associated to mathematical relations is rarely assumed or realized by the students. Thus, by solving equations, one "applies" mathematics which is regarded as an abstract system. The content or the situation are at first absent and they are expected to arise through applications. In other words, the contextual richness or depth will be added to the mathematical concepts with the application exercises. Mathematical notions (more today than in the past) are mainly determined by a kind of inter-conceptual organization. For instance, the notion of variable has become a special sort of cartesian product. The science teachers as the"appliers" or the utilizers make use of notions that are, so to speak, bestowed by the mathematics teachers. However, even though this epistemological perspective is never challenged, the day-to-day "life" in schools is not so simple. It frequently happens, for instance, that science teachers express concerns because mathematics teachers have not quite "prepared" the "right" object. It is well known that students complain that they have to deal with the functions of their mathematics teachers and that of their science instructors. This is also too often the case for vectors or logarithmic functions.

Several examples of contextualized mathematics can be provided. The reader may refer to Janvier (1989). They all allow us to re-examine the notion of application.

Application versus Contextualization or Modelization Revisited.

It is worth comparing "applying mathematics" with the process of modeling in science. Both start from the phenonenon which is at first examined in order to find out patterns, relations already known to be extended...This first stage leads to the formulation of a mathematical equivalent counter-part of the situation, which is called a model, and that will stand for the situation as the analysis will be carried on. In fact, the model belongs to another level or mode of representation and imply necessarily a selective reduction of the factors involved as more fully expounded in Janvier (1980). This is what makes it abstract. In the case of application, the real elaboration of an abstract model consists much more in a selection of the right equations or relations. But, as the figure 2a illustrates it, the dramatic similarity between application and modeling is that at some point the entire work is assumed to be carried out within the model or within the mathematical domain. The interpretation process in modelization is aimed at checking the domain of validity of the model and eventually brings about a new "formulation-interpretation" cycle yielding a more refined model. For applications, going back to the context enables one to detect calculation mistakes or inappropriate selection or processing of equations. The article challenges the fact that always, at some point, work is exclusively achieved in the mathematics domain. For some efficient users, the context remains present and this fact entails particular abilities.

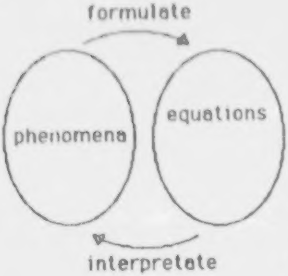

figure 2a.
In applications, the work is done in the "mathematics domain"

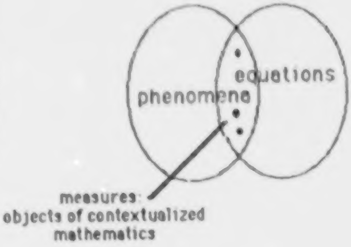

figure 2b.
Contextualization provides an overlap as "working space".

This position is suggested by the research whose results were exposed above and several examples I have analyzed. The notion of **contextualization** has been introduced to convey the idea that the context is kept into play even though the appearance would allude to processes being achieved solely in the mathematics domain. The modes through which the context exerts its influence are diverse: diagrams, graphs, verbal descriptions, mental images, imagined or actual actions. Their role in the process of contextualization is to bring closer the situation and the "mathematics domain" (see figure 2b.). The overlap being produced becomes the "working space" of contextualized mathematics whose objects are quantities, measures or magnitudes..They consist in mathematical ideas that have kept some concrete connotations or in other words of mathematical entities not entirely stripped of their situational content. The cognitive status of those "quantities" have clearly to be examined further mainly in their subtle and implicit support for the reasoning in problem solving settings.

The consequences for research.

The project we are about to start makes the **assumption** that day-to-day usage by ingeneers and technician induces particular procedures. On the one hand, there exist particular mathematical procedures that are used; but on the other hand, they are context related in the sense that the mathematical notions need, in order to be mentally worked on ,the interventions of some features borrowed from the context. **It is aimed at verifying the nature of the contextualized mathematics** some electrical engineers and technicians resort to while solving circuit problems.

In fact, the way they use mathematics will be described as a form of **coordination between representations** as mentionded in Kaput, Goldin Lesh and Janvier. In other words, it will be envisaged as a form of **translation** that does not simply imply **going from one source to another**, but also **coordinating both sources** taking into account the fact that the connotations attached to the concepts are present. Consequenly, the notions of primitive conception (Janvier), phenomenological primitive (diSessa), mental models (Gentner), spontaneous reasoning (Viennot) will be fondamental for our analysis in that they contain, I believe, the basic ingredients of contextualized mathematics. The notion of theorem-in-action of Vergnaud seems to be equally inspiring. A major step in the research will be to search deeper into the work of Joshua (1982), Closset (1983) and others to find the basic electricity models from which the ones observed will be related.

As for the observation techniques, we shall make use of a scheme in which the subjects will have to inter-act two by two on a rich set of situations

Bibliography

Carraher, T. N., Carraher, D. W., & Schliemann, A. D.(1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology* . 3, 21-29.

Carraher, T. N. & Schliemann, A. D.(1987). Manipulating equivalences in the market and in mathematics. In *Proceedings of the 11th conference of P.M.E.* Montréal. July

Carraher, T. N., Carraher, D. W., & Schliemann, A. D.(1987). Written and Oral Mathematics. *Journal for Research in Mathematics Education.* Vol 18-2. March.

Closset, J.-L. (1983). *Le raisonnement séquentiel en électrocinétique.* Thèse de doctorat de 3ème cycle. Université de Paris VII.

Gentner,D. and Stevens, A. L. eds. (1983). *Mental Models.* Lawrence Earlbaum Associates.

Girardon, C., Janvier, C. (1987). Conflictual conceptions of transformations. *Proceedings of the P.M.E. XI Conference*, vol. III, pp. 148-154, Montréal, juillet.

Freudenthal, H. (1983) Didactical phenomenology of mathematical structures. Reidel Publishing Co. Dordrecht.

Janvier, C. eds. (1987). *Problems of representation in the teaching and learning of mathematics.* Lawrence Earlbaum Associates.

Janvier, C. (1989). Contextualization and Mathematics for All. Accepted for publication in the *1989 N.C.T.M. Yearbook* .

Joshua, S.(1982). L'utilisation de schéma en électrocinétique: aspects perceptifs et aspects conceptuels. Proposition pour l'introduction de la notion de potentiel en électrocinétique. Resnick, L. (1987). Learning in School and Out. *Educational Researcher.* December.

Vergnaud, Gérard(1979). Multiplicative structures. in Lesh and Landau (ed.): *Acquisition of mathematics concepts and processes.* Academic Press. 1979.

Viennot, L. (1979). *Le raisonnement spontané en dynamique élémentaire.* Hermann. Paris .

TO *INCULCATE* VERSUS TO *ELICIT* KNOWLEDGE

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"The teacher's dilemma is to have to inculcate knowledge while apparently eliciting it." [Edwards and Mercer 1987]

It is usual that a mathematics teacher is required by some syllabus, scheme of work, or curriculum to teach stated mathematical concepts to pupils. As a result of the teaching/learning situation pupils acquire certain knowledge. A constructivist view of learning suggests that pupils learn as a result of their own construal affecting their own experiential world, which implies that any inculcation *of knowledge can only be successful if it contacts the experience of the learner. The dilemma for the teacher is* ,"I've got to get them to construe x"

This report concerns teaching approaches which seek to encourage effective construal by the learner of required mathematical concepts. It includes extracts from a case study of one classroom where there has been evidence of success in methods used to elicit *knowledge rather than a dependence on* inculcating *it.*

Successful teaching of mathematics involves a teacher in intentionally and effectively assisting pupils to construe, or make sense of mathematical topics. There are many words and metaphors used to describe the process by which a teacher teaches and pupils, as a result, gain knowledge. A familiar one is that of inculcation, of *giving* or *handing over* knowledge. The *giving of a good explanation* carries with it a sense of one person (the teacher) successfully transmitting to another person (the pupil) some item of knowledge. Although some teachers and pupils still see successful teaching in terms of such transmission, there is a movement towards belief that in and of itself this is not enough. The literature which relates a *constructivist* view of learning to classroom teaching of mathematics suggests that teaching has to take into account individual construal and its relation to and modification of individual experience in the learner (see for example von Glasersfeld (1983), Kilpatrick (1987), Cobb (1988), Jaworski (1988)). The view of knowledge and of learning which this philosophy promotes may seem at variance with the requirements which society places on its educational system, often through legislation, in terms of learning being measured by the ability to reproduce certain predefined items of knowledge on demand. This paper seeks to highlight some of the issues in teaching for effective learning, particularly with regard to such requirements, and reports on the work of one teacher who aims to meet the requirements while working in a way he believes to be most fruitful in providing opportunity for pupils to make sense of mathematics.

Teaching approaches

In a desire to make teaching more relevant to the learner, the phrase 'learning from experience' has gained some currency, and this, like the transmission metaphor, has proved inadequate in describing a basis for effective learning. In their book Common Knowledge, Edwards and Mercer (1987) report on a number of lessons on 'pendulums' which were taught by a teacher who, it was reported, believed in the importance of *experiential learning*. Briefly, this suggests that pupils can best learn a concept when they have experienced for themselves manifestations of that concept. Thus the teacher took care to provide opportunity for pupils to experience aspects of pendulums, and to explore some of their properties. Implicit here seemed to be the idea that experience leads to learning. The teacher in providing the experience is therefore promoting learning.

Driver (1983) comments, of such experimental work in science teaching,

> 'Activity by itself is not enough. It is the sense that is made of it that matters.'

She claimed, of science lessons, that often a lesson ends with the clearing up after practical work is finished, so that opportunity for discussion of how experiences relate to new ideas is missed. Paul Cobb, who has worked extensively with teachers on the implications of a constructivist philosophy for the mathematics classroom, was asked how he would reply to the question from a teacher,

> "If I leave pupils to construct for themselves, how can I be sure that they will construct what I want them to construct?"

He said (Paul Cobb 1988),

> The idea that we give the children some blocks or some materials and we leave them alone, and we come back in fifteen years' time and expect them to have invented calculus just makes absolutely no sense whatsoever. The teacher is still very much an authority in the classroom. The teacher still *teaches*.

Edwards and Mercer's pendulums teacher did not leave pupils just to come to their own conclusions as a result of their experimentation with the pendulums apparatus. She engaged with them in extensive discussion about the principles which were involved. Nevertheless, conclusions were drawn that these pupils' knowledge, or understanding, of pendulums was still deficient. The authors distinguish between *ritual* knowledge and *principled* knowledge. They quote from research by Taba and Elzey (1964), citing the instance of a girl who regularly achieved good marks in mathematics and described her procedures as follows:

> "I know what to do by looking at the examples. If there are only two numbers, I subtract. If there are lots of numbers I add. If there are just two numbers and one is smaller than the other it is a hard problem. I divide to see if it comes out even and if it doesn't I multiply."

Edwards and Mercer comment,

> 'What we are calling *ritual* knowledge is a particular sort of procedural knowledge, knowing how to do something. In many contexts, of course, procedural knowledge is entirely appropriate and exactly what is required. This was the case with learning to do clay pottery, and was also an important part of the lessons on pendulums; the pupils had to know how to operate their apparatus, their stop watches and calculators, and much of their ability to get through the lessons required knowledge which was essentially procedural. Procedural knowledge becomes 'ritual' where it substitutes for an understanding of underlying principles. Ritual knowledge is the sort exhibited rather crudely by the pupil in Taba and Elzey's example, ... *Principled* knowledge is defined as essentially explanatory, oriented towards an understanding of how procedures and processes work, of why certain conclusions are necessary or valid, rather than being arbitrary things to say because they seem to please the teacher.

They go on to discuss the pendulum lessons from the point of view of the *principled* knowledge which pupils gain and conclude that despite the teacher's declared attempts to enable a principled understanding of the operation of pendulums, nevertheless what they observe is only a ritualistic parroting of the aspects of pendulums which the teacher has emphasised during their experimental work.

It is *what* is done which seems to be crucial. The pendulums teacher, in Edwards and Mercer's research prompted pupils in various ways and they appeared to sieze on her cues as the important pieces of knowledge which they were expected to take from the lessons. The authors suggest that these pupils were not encouraged to conceptualise pendulums adequately. However, it is easy for an observer to make judgements about teaching which appears inadequate because understanding appears ritualised, but very much harder to identify teaching strategies which lead to successful principled understanding. Many teachers and researchers have discussed activities and strategies which have been designed and demonstrated to promote mathematical thinking in pupils in the classroom (see for example Brown and Walters (1983); Collins (1988); Jaworski (1985); Cobb, Wood and Yackel (in press)). In all of these, pupils are observed to engage with mathematical thinking and, it is suggested, to take on some of the responsibility for their own learning, in that they are no longer simply seeking for teachers' explanations. In classess which I have observed as part of my own study (See Jaworski (1988) for details and methodology) it has been possible to observe pupils engaged in their own mathematical activity, actively constructing mathematics themselves. The knowledge that is gained by pupils in consequence of this might be described as *principled*. What is often difficult for the teacher in such circumstances is to assess pupils' thinking in terms of the standard topics which the curriculum requires, and moreover to ensure that the thinking includes ability to succeed in standardised tests on these topics. Put into the context of pendulums, what teaching approaches would result in a *principled* understanding of pendulums and would, as well, enable pupils to succeed in standard tests regarding pendulum operation?

Teaching requirements

In my own research, one of the teachers whom I observed planned a series of lessons on 'tessellation' for a class of eleven-year-olds. During the planning she stated that she wanted pupils to *investigate* aspects of tessellating polygons. A number of activities were designed in which pupils took part. I observed lively discussion about *which* polygons would tessellate - for example, some pupils decided that regular pentagons would never tessellate, whereas quadrilaterals would *'if you could draw them better'*. They had experimented by drawing their own shapes, cutting them out and fitting them together, but recognised that their shapes were imperfect and that they had to take this into account. Despite a certain amount of what she later referred to as *prompting*, the teacher felt that pupils had not gone as far with ideas as she would have liked them to, particularly in consideration of angles in the polygons. She said,

> The group work that they did, I'm not sure that it worked exactly as I'd hoped it would work or that they actually focused on the angles meeting at a point as I hoped they might .... They kept referring to the fact that if they were able to make the shapes into quadrilaterals or rectangles, that they would be able to tessellate the shapes. But yet they weren't all convinced that all quadrilaterals tessellated. That was the thing I wanted them to go on to.

She had been pleased with the mathematical activity and discussion which provided insights into the sense which the pupils were making, but she wanted more. She was justifiably influenced by her syllabus demands, and unsure about how she was going to fulfill their requirements. Her view of investigational work involved only a minimal level of intervention, and she was in the process of reconsidering what that intervention might involve.

Teachers have to take account of the educational system in which the teaching takes place, the requirements of this system and the expectations of the consumers of the system. Currently the British government is in the process of introducing legislation to establish a National Curriculum in schools in England and Wales. This will have associated attainment targets and pupils will be tested nationally at the ages of 7, 11 and 14. The form which attainment targets and testing will take is the subject of much controversy. It is sometimes implied that it is a straightforward matter to say what standard a particular pupil has reached at any stage. One attainment target , for example, requires that pupils can

> Multiply a 3-digit number by a 2-digit number and divide a 3-digit number by a 2-digit number in both cases without a calculator. (NCC 1988 - Target 3, level 5)

How this will be tested is not yet clear, but if testing involves asking pupils to apply the algorithm to particular given numbers, then teachers will want to ensure that pupils are able to tackle this successfully. It raises questions about what methods of teaching are appropriate to such success, and indeed what is implied for the childs overall mathematical development.

One of the dangers of a rigid curriculum and system of testing is that teachers, pressured by perceived expectations and shortage of time, feel unable to exploit teaching methods which require a high degree of confidence for their application. This confidence lies in the belief that the methods will promote learning most successfully, so that the requirements of the system will be fulfilled along with other learning objectives. In terms of the attainment target quoted above, it might mean that as children are helped to make sense of the arithmetic operations of multiplication and division generally, they will learn to cope with the algorithms required. The teacher on tessellation however, felt that her methods had been unsuccessful, because she had not been able to elicit the particular mathematical ideas which she wanted from pupils as a result of the activities she had provided. Edwards and Mercer, on the other hand, felt that the pendulums teacher had sacrificed a principled understanding of the pedulum concepts in her pupils because of certain results which she had been at pains to inculcate. In the remainder of this paper I shall quote from a case study of one teacher whose lessons I observed over a period of six months. This teacher was very confident in his teaching methods, which might be described as *investigative* in style, and believed in trying to teach in a way which provided pupils with the best opportunity for learning mathematics.

From a case study of an investigative method of teaching.

An investigative method of teaching, briefly, involves encouraging pupils to explore ideas and to develop their analytical and problem solving abilities. My own research in its global sense aims to characterise such teaching, and this case study is just a part of it. (See Jaworski (1988) for excerpts from another such case study.) The teacher, Ben, teaches in a 12-16 comprehensive school where he is head of mathematics. I observed him teaching a fourth year class which he had been teaching for just over one year. They were starting preparation for GCSE (General Certificate of Secondary Education) assessment at the end of their fifth year. This involves continuous assessment of course work and a final examination. I shall give examples of a way of working which encouraged the development of a *principled* mathematical understanding, while keeping in sight the demands of the final examination and its importance for pupils.

I observed a lesson where pupils were exploring Kathy shapes - Ben's name for shapes which have the same area as perimeter. At the start of the lesson, Ben had asked for silence and said, "Let's recap last lesson". Pupils responded variously with reference to areas and perimeters of various shapes and to *Kathy shapes*. Ben asked, "What *is* a Kathy shape?" Some negotiation led to an articulation of '*same area as perimeter*'. Ben pushed them to explain what they meant by area and perimeter, and a number of pupils joined in struggling to express their understanding of these terms. Ben commonly used this approach to a follow-up lesson, beginning with whole class reconstruction of ideas to draw pupils back into mathematical thinking. On this occasion, different groups in the room had chosen to work on different shapes, some on squares and rectangles, some on triangles, some on circles, others more ambitiously on polygons generally. Ben had told me before the lesson that they would work in groups of their own choice, and that this choice might reflect their own level of ability. In a subsequent lesson, he said similarly, "How do you want to work, pairs, groups, ...?" A brief period of class negotiation followed where pupils decided what they would start working on, and so moved into like minded groups. I observed that Ben rarely constructed groups himself and he later commented that he wanted pupils to make decisions about how it was most appropriate for them to work. If he felt strongly that anyone was making the wrong decision, then he would suggest why they might do otherwise - for example, one very bright pair of boys were advised not to work together because they vied with each other in a way which Ben felt was not helping them to make progress.

Three girls had chosen to look for Kathy *triangles*.. They had started by drawing an equilateral triangle of side two units whose *height* , they claimed, was *two* units. "What do you mean by *height* ?", said Ben. (Had they confused height with the length of a side?) One of them traced out the vertical height with her pencil. It looked as if she understood *height*, but how could she think it was of length *two*? Another girl started to draw the triangle accurately and when complete she measured its height. It was less than two! While she was drawing, the others, at Ben's prompting, discussed what the height should be. They first thought that it should be the same as the sides of the triangle, and then that it would be *more* than that. They were surprised when it turned out to be less than two. Ben suggested that they should draw other triangles and compare heights with sides. "What am I always saying?", he said. "Is there a pattern?", one girl replied.

At some point early in our discussions, Ben had said to me, "You should ask them (the pupils) what it is that I always tell them to do." He was quite confident as to the reply that I should get. It emerged that the instruction was *look for a pattern*, or *is there a pattern?*, and indeed whenever there was a hint of "what questions should we be asking?", someone in the class came up with "is there a pattern?", (sometimes when, to Ben's chagrin and my amusement, it was inappropriate!). The words symbolised Ben's belief in mathematics being about expressing generality. Implicit in the interchange here was emphasis on conjecturing, on trying out special cases, and on seeking for generality. As I continued to observe the class I saw more and more evidence of pupils' intuitive appreciation of these processes. For example, one group in the class had found a *Kathy square* - a square of side four units. In trying to explain, one boy said, "If you times 4 by 4 you get 16, and then if you times 4 by itself you get sixteen." Ben replied, "I don't understand the difference?". He said that the two things sounded the same, so what did the boy mean? The conversation proceeded:

> **Boy**: Pick a number. **Ben:** Three. **Boy:** Right, what is three times by itself? **Ben:** Nine. **Boy:** Because a square has got four side, to find the perimeter you have to times that, three times four. **Ben:** Twelve. **Boy**: But if you do it with four, they both equal the *same* number, sixteen.

As a response to Ben's deliberate provocation, the boy had used a generic example to demonstrate a general understanding of Kathy squares.

In discussion with me after the lesson Ben referred to the importance of *recognising* particular misconceptions which pupils have, in order to try to correct them. A number of misconceptions had been evident, for example the one about the height of a triangle described above. Another involved a belief by some pupils that *any* three numbers which they might choose could be lengths of the sides of a triangle. Yet another involved confusion between the terms *area* and *perimeter*. Ben admitted to surprise as to the difficulty which pupils had had in recognising the vertical height of a triangle, and estimating its length. He felt that from work done previously, pupils would have had a better understanding of these concepts. Yet he was happy that an opportunity had arisen in which pupils could tackle them. He felt that the situation in this lesson had aided his recognition of these misconceptions better than a more narrowly defined activity might have done.

In a subsequent lesson Ben returned to Kathy shapes. Some work had been done on Pythagoras' theorem and he expected that use of the Pythagorean result would help in the search for Kathy *triangles*. After the lesson I reminded Ben that he had expected pupils to use Pythagoras to calculate lengths in triangles, and asked whether he felt this had happened. His reply was,

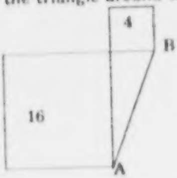
> I suddenly realised that it wasn't. I suddenly saw people measuring and I was going to jump in and say "hang on, why aren't you calculating it?", and then I realised, if you're going to do it roughly why not measure it to hunt it down, you know, a far better strategy ... . I wanted them to use Pythagoras and they've actually come up with a better strategy. And I then made a decision that their's was a better one than mine and they might as well use it ... why force people to use inefficient systems? But I do realise that when they actually come down and say, "I've measured it", I can turn round and say, "but is it exact?", then they have to start using Pythagoras.

Ben was ready to admit that often what occurred in a lesson was different to what he had planned because he believed that flexibility in following up pupils' methods and ideas was important. We talked about where the work might go from here. He referred to one group who had been using *graphs* to hunt down Kathy shapes. He felt that it might be helpful to others in the class to work on their graphical method, as well as providing some context for graphical work required by the GCSE syllabus. In talking about how he made decisions for a particular lesson he said,

> If I actually do graphs through Kathy shapes, will they be bored with Kathy shapes then or not? If they're bored with Kathy shapes, I'm going to lose out; I'm not going to get my mathematical points over. So you weigh things like that up - rule of thumb methods - I have no fixed techniques, I just get a feeling.

In the event the class did work on graphs of Kathy shapes. The activity provided new opportunities for pupils to express their understanding of area and perimter, and a familiar context in which to develop ideas of graphical representation. It is tempting to say that at this stage most pupils were reaching a principled understanding of many aspects of area and perimeter. Justification of this would need a closer analysis of what was done and said.

However, one instance of the development of principled understanding is worth reporting. In a lesson on vectors I observed two boys, Lee and his partner, Danny. They had been asked to invent some vectors of their own and work out the lengths of the vectors. Lee explained to Danny what he thought they had to do. He wrote down the vector AB, as below, placed points A



$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

and B on a grid, drew the triangle around them, drew squares on two sides of the triangle as shown, wrote the square numbers in the squares, and then worked out mentally aloud: " 16 plus 4, that's 20, square root ... about 4.5". (He demonstrated considerable skill in estimating square roots.) Danny seemed to follow what he had done, and the pair set about independently inventing vectors and finding lengths. In each case Lee drew a diagram similar to the one above, writing the square numbers into the squares. He then performed the calculation mentally and wrote down the result. He seemed to demonstrate a fluency with using the Pythagorean result, but I wondered what role the diagrams were playing for him in its use. I mentioned this to Ben after the lesson. He reminded me that in one of the lessons on Kathy shapes, Lee had had difficulty with the use of Pythagoras, and had certainly shown no fluency with it. He was delighted to hear that Lee seemed to have 'learned' its use in the intervening period. I wondered if Lee would in fact be able to abstract it - whether he would be able to cope without his diagrams. With Ben's agreement, in another lesson (where in fact Pythagoras was not in use at all) I went over to Lee and asked if I could try something out on him. He seemed agreeable, so I asked if he remembered finding lengths of vectors, and if he would work out the length of a vector for me. I gave him the vector numerically, (5,7). He drew a diagram similar to the one above, wrote the numbers 25 and 49 in the squares, then did a mental calculation resulting in looking for the square root of 74. I then asked if he would try another but without drawing a diagram. He worked it through aloud, getting to the result with hardly a hesitation. Here, it seemed, was an example of learning actually having occurred over a period of time. I was reminded of a saying of my colleague John Mason, that 'teaching takes place *in* time but learning happens *over* time'. Lee seemed to have demonstrated a principled understanding of the Pythagorean result, and it seemed likely that he would be able to reproduce it for use in an examination question. One of the problems of evaluating investigative styles of teaching is that formal testing can be inappropriate. It is often not clear what you actually test. What is required is a sense of the mathematical development which has taken place for pupils and, as this is different for every pupil it is very hard to measure. The teacher working in this way has to develop ways of perceiving the progress which individual pupils make, and this example of Lee was one of the treasures which a teacher hopes for.

At the end of one lesson, two boys remained after the rest of the class had left, in conversation with Ben and myself about some aspect of the lesson. The conversation turned onto aspects of teaching and learning which the boys thought were important. Some of the words from one of the boys go some way towards a vindication of the methods which this teacher used, and their perceived success:

> To tell you the truth (addressed to me, although Ben was present) I mean, Mr xxxx's ... a different kind of teacher completely. Before, you've had sums that you've been set .... At first, to tell you the truth, I didn't like him as a teacher. I thought, "No. Pathetic!", - you know, "this isn't maths - what's this got to do

> with maths?" And as I've come along I've realised that it's got a lot to do with maths. To have to learn rather than just have to sit and "Oh, I've done 50 sums today.", "*I've* done a hundred. You don't bother about that now, you just concentrate, and at the end of a lesson you've learned something - ... I've really progressed.

Ben was nevertheless concerned that his class could cope with the externally set GCSE examination papers in which pupils had to individually tackle questions on mathematical topics from the GCSE syllabus. He set the class one past paper during the time that I was observing, and reported that he was satisfied with pupils' results at this stage of their course. What was particularly interesting for me was how he tackled errors in pupils' solutions. Rather than produce a set of model solutions, he duplicated a set of *incorrect* solutions. Pupils had to compare their own solutions, right or wrong, with the incorrect solutions given. Their task was to decide wherein lay the errors, and to work together to correct them. Discussions which I heard indicated that pupils were being challenged to reconsider their understanding of concepts involved through the various solutions which they had to compare.

In conclusion, I believe that I was given repeated evidence of the development of principled understanding of mathematical concepts in the pupils I observed. I saw a teacher who promoted independance of pupils in his classroom, and encouraged pupils to take responsibility for their own learning, while maintaining a concern for concept development, and recognition and tackling of misconceptions. He was working to a standard syllabus, and there was evidence of pupils' mastery of standard topics on this syllabus. It remains to be seen how the pupils will fare in the standardised testing at the end of their course.

References

Brown S.I. and Walter M.I. (1983) The Art of Problem Posing. Hillsdale, NJ: Lawrence Erlbaum Associates

Cobb P. (1988) The Tension between Theories of Learning and Instruction in Mathematics Education In Educational Psychologist 23(2), 87-103. Lawrence Erlbaum Associates

Cobb P. (1989) Interview in Block I Audio Cassette : ME234 Using Mathematical Thinking Milton Keynes: Open University Press.

Cobb P. Wood T. and Yackel E (in press) A Constructivist Approach to Second Grade Mathematics . In von Glasersfeld E. (Ed.) Constructivism in Mathematics Education. Dordrecht, Holland: Reidel

Collins A (1988) Different Goals of Enquiry Teaching In Questioning Exchange, 1988, Vol 2 No 1 pp39-45

Driver R. (1983) The pupil as Scientist. Milton Keynes: Open University Press

Edwards D. & Mercer N. (1987) Common Knowledge London : Methuen

Jaworski B. (1985) A poster lesson. In Mathematics Teaching 113

Jaworski B. (1988) One Mathematics Teacher in PME XII proceedings Vesprem, Hungary : OOK

Jaworski B. (1988) Is versus Seeing As : Constructivism and the mathematics classroom. In Pimm D. (Ed) Mathematics Teachers and Children. London: Hodder and Stoughton

National Curriculum Council (1988) Consultation Report Mathematics London :NCC

Taba H. & Elzey F. (1964) Teaching strategies and thought processes Teachers College Record, 65, 524-534

von Glasersfeld E. (1983) Learning as a Constructive Activity in PME - NA proceedings Montreal.

VAN HIELE LEVELS AND THE SOLO TAXONOMY

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A comparison is made between van Hiele levels of development in geometry and the Structure of Learning Outcomes (SOLO) taxonomy. It is hypothesized that if the correspondence between van Hiele levels and SOLO levels is tenable then the latter may be used as an operational scheme for characterizing a posteriori the learning outcomes in geometrical tasks without going through the process of identifying van Hiele level indicators empirically, deriving them for a multitude of geometrical tasks. This is particularly relevant since the construct validity of the SOLO taxonomy has been established. The logical comparison and an illustrative example revealed a high degree of similarity between the two sets of levels.

A number of studies have focussed on van Hiele model of development in geometry. One line of investigation was to establish the hierarchical nature of van Hiele levels. Another line was to characterize van Hiele levels operationally using student behaviors as level indicators. Conclusions in this area may be summarized as follows:

1. In general there is evidence in support of the hierarchical nature of van Hiele levels (Mayberry, 1983).
2. Students can be assigned a van Hiele level based on their performance on geometrical tasks. However, the decalage phenomenon was observed across different tasks supposedly in the same van Hiele level.

Moreover, some students performed below the expected van Hiele level (Burger and Shaughnessy, 1986; Mayberry, 1983; and Usiskin, 1982).

An assumption underlying van Hiele model and subsequent studies is the existence of hierarchical levels, each of which has its idiosyncratic mode of functioning and can be characterized by its own set of developmental tasks. Student behaviors on such tasks may be determined empirically. Assignment to van Hiele levels on each of these tasks may be made by utilizing the student behaviors as van Hiele level indicators. This seems to be reminiscent of Piagetian stages of development.

An alternative scheme for studying performance on geometrical tasks is classifying learning outcomes by looking at their structure rather than classifying individuals by looking at indicators of some cognitive abilities. This is not to suggest to ignore van Hiele levels but rather to look at them as cognitive abilities which reflect typical modes of functioning predominant at different stages of development in geometrical thought. The Structure of the Learned Outcomes (SOLO) taxonomy developed by Biggs and Collis (1982) is such a scheme. In this taxonomy, the structure of the learned outcome occurs within each mode of functioning. The learned outcome becomes increasingly complex but structurally the complexities at each mode are the same. On the other hand, the van Hiele levels which are specifically developed to describe geometrical thought assume a sequence of cognitive abilities characterizing a sequence of developmental stages. The descriptions of *SOLO* and van Hiele appear in Figure 1.

Purpose

The purpose of this paper is twofold. First, to establish logically the correspondence among van Hiele levels ar the levels in the SOLO taxonomy. Second, to present an analysis of an illustrative example to demonstrate the results of the logical analysis.

If the correspondence between van Hiele model and SOLO taxonomy is tenable, then the SOLO taxonomy will provide an operational scheme for characterizing a posteriori the learning outcomes in geometrical tasks without going through the process of identifying van Hiele level indicators by empirically deriving them for a multitude of geometrical tasks. Moreover, the construct validity of the SOLO taxonomy has been established in terms of the hierarchical nature of its levels and in terms of partioning students at different age levels into interpretable groups that reflect the SOLO levels (Romberg, Jurdak, Collis and Buchanan, (1982). In addition, the usefulness of the SOLO taxonomy in assessing levels of reasoning in mathematical problem solving has been established (Collis, Romberg, Jurdak, 1986). At last, if the correspondence is established, then the SOLO level which is matched with the van Hiele level will be hypothetically the predominant level of reasoning in that particular van Hiele level.

Logical Correspondence

Figure 1 presents the van Hiele levels with the hypothesized corresponding SOLO levels. Column 1 shows the typical van Hiele levels with their descriptions as they appeared in

1 Van Hiele Level	2 SOLO Level
	Prestructural. Response represents the use of no relevant aspect.
Level 0 (Visualization). Visual consideration of the concept as a whole but not its properties.	Unistructural. Response represents the use of one relevant aspect.
Level 1 (Analysis).Perception of properties of geometric properties but these properties are isolated and unrelated.	Multistructural. Response represents the use of several disjoint aspects.
Level 2 (Abstraction). Perception of relationships between properties to form abstract definitions.	Relational. Response represents the use of all aspects related into an integrated whole.
Level 3 (Deduction). Reasoning within the context of a mathematical system using deduction, anxioms, and definitions.	Extended Abstract. Comprehensive use of all relevant aspects together with related hypothetical constructs and abstract principles.
Level 4 (Rigor) Comparing systems based on different axioms in the absence of concrete models.	

Figure 1. A comparision between van Hiele and SOLO levels.

the literature (Hoffer, 1981; Mayberry, 1983; and Burger and Shaughnessy, 1986). Column 2 shows the SOLO taxonomic levels with their descriptions using sources like Biggs and Collis (1982) and Collis, Romberg, and Jurdak (1986).

A careful study of the descriptions of the van Hiele and SOLO levels in Figure 1 reveals that, with the exception of two levels, the two sets may be reasonably matched. This means that the classification of a response of a geometric task belonging to a particular van Hiele level falls within the corresponding SOLO level. In other words, when the SOLO taxonomy (which is of general nature) is applied to geometric tasks, the SOLO levels of reasoning are very similar to van Hiele levels (which are specific to geometric thought). However, there seems to be two exceptions. First, the prestructural SOLO level has no corresponding level in van Hiele model. This is understandable since the prestructural SOLO level is simply a refusal or inability to become engaged in the task. Second, van Hiele level 4 (rigor) has no corresponding SOLO level. It is to be noted here that van Hiele level 4 has not been identified for pre-university students and rarely for math major students.

An Illustrative Example

To illustrate the correspondence between SOLO and van Hiele levels, an example from the literature on van Hiele model (Burger and Shaughnessy, 1986) was classified in accordance with the SOLO taxonomy and compared with its van Hiele classification. The example was taken from a study by Burger and Shaughnessy (1986) in which many geometric tasks were administered to a sample of students using the interview method.

The purpose of the study was to characterize the van Hiele levels operationally by students behaviors. The task is called "Identifying and Defining task" and it consisted of identifying quadrilaterals from a given sheet of drawn quadrilaterals by putting S on each square, R on each rectangle, P on each parallelogram and B on each rhombus. The defining task was to basically define the figure by giving a minimal list for characterizing each figure (necessary and sufficient conditions). The responses of students were taped, analyzed and assigned a van Hiele level by three reviewers.

In the SOLO taxonomy framework, the relevant data in the task consist of the quadrilaterals and their properties as reflected in the drawings. The increased use of data and relationships result in an increased structural complexity in the response.

Figure 2 shows a detailed comparative classifications of the "Identifying and Defining Task" in SOLO and van Hiele models. The analysis in Figure 2 provides support for the possibility of matching the van H[illegible]d SOLO levels.

Concluding Remarks

There seems to be theoretical as well as empirical support for the possibility of matching the SOLO and van Hiele levels. Thus, responses to geometric tasks can be characterized in a way compatible with van Hiele levels without necessarily deriving the student behaviors indicative of van Hiele levels for every geometric task. The assumption of SOLO taxonomy is that the structural levels of responses occur within each development stage, thus the extended abstract level in one

Response +	SOLO Classification	Van Hiele Classification.+
• identified some quadrilaterals but failed to identify others of the same kind because of consideration of irrelevant data.	• Unistructural.Used one relevant aspect (the figure).	Level 0
• identified types of quadrilaterals but without class inclusion.	• Multistructural. Used several relevant disjoint aspects (the quadrilateral and its separate properties).	Level 1
• identified quadrilaterals correctly and defined them by their components.	• Relational. Used all relevant information and the relationships among them (the quadrilateral, properties of its components, sufficient conditions to define the shape).	Level 2
• defined various quadrilaterals independantly of each other then checked definitions to make sure that they permitted the desired class inclusion.	• Extended abstract. Comprehensive use of the given information (figure, properties, relations) with related hypothetical constructs and abstract principles (tested on the data).	Level 3

Figure 2. Comparative classification of the (Identifying and Defining Task" in SOLO and van Hiele models.+ Source: Burger and Shaughnessy (1982).

stage becomes the unistructural level in the subsequent stage to form a new learning cycle. Consequently, the predominant level of reasoning in a particular van Hiele level would be characterized by the corresponding SOLO level. The possible occurence of all SOLO levels within a van Hiele level, with the predominance of one of them, probably explains the decalage phenomenon onserved in van Hiele levels.

References

Biggs, J.B., and Collis, K.K. (1982). Evaluating the quality of learning: the SOLO taxonomy (structure of the observed learning outcome). New York: Academic Press.

Burger, W.F., and Shaughnessy, J.M. (1986). Characterizing the van Hiele levels of development in geometry. Journal for Research in Mathematics Education, 17, 31-48.

Collis, K.F., Romberg, T.A. and Jurdak, M.E. (1986). A technique for assessing mathematical problem-solving ability. Journal for Research in Mathematics Education, 17, 206-221

Hoffer, A. (1981). Geometry is more than proof. Mathematics Teacher, 74, 11-18.

Mayberry, J. (1983). The van Hiele levels of geometric thought in undergraduate preservic teachers. Journal for Research in Mathematics Education, 14, 58-69.

Romberg, J.A., Jurdak, M.E. Collis, K.F. and Buchanan, A.E. (1982). A report on the NIE/ECS Item Development Project: Construct validity of a set of mathematical superitems. Madison: Wisconsin Center for Education Research.

Usiskin, Z. (1982). Van Hiele levels and achievement in secondary geometry. ERIC Document Reproduction Service No. ED 220 288.

A PERSPECTIVE ON ALGEBRAIC THINKING

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This theoretical paper examines the issue of algebraic thinking. The Research Agenda Conference on the Learning and Teaching of Algebra pointed to the topic of algebraic thinking as an area sorely in need of research attention. Since so little discussion of the topic has taken place, no real consensus exists as to what algebraic thinking means. This paper argues for a particular interpretation of the phrase and then goes on to document findings from some of the few studies related to this research issue.

One of the topics pointed to in the Research Agenda, an outcome of the 1987 Research Agenda Conference in Algebra (Wagner & Kieran, 1989) as an area sorely in need of research attention is that of algebraic thinking. Some of the questions raised by conference participants were:

- What dimensions of algebraic thinking can we identify (e.g., knowledge of structures, use of variables, understanding of functions, symbol facility/flexibility, generalizing, inverting and reversing operations and relations, ability to formalize arithmetic patterns, etc.)?
    - a. What kinds of thought processes are involved in various algebraic topics?
    - b. What kinds of thinking processes are required to apply algebra to problem situations?
    - c. What are the effects of studying specific topics on students' facility in algebraic thinking?
- How does/can a given dimension of algebraic thinking develop?
    - a. What skills/concepts mediate algebraic thinking?
    - b. What instructional strategies promote the development of certain dimensions of algebraic thinking?

c. Are there particular types of (word) problems that stimulate the development of algebraic reasoning?

All of these are interesting, researchable questions; however, there would appear to be something missing in the above list. That something is a meaning for algebraic thinking--a working definition or characterization of the phrase that might remove some of the ambiguity from the above questions. In this paper, I will attempt to argue for a particular interpretation of what we might consider algebraic thinking to be, and then provide some research evidence documenting students' difficulty with this aspect of algebra.

Love (1986) has proposed the following chracterization of algebraic thinking:

> Algebra is now not merely "giving meaning to the symbols," but another level beyond that; concerning itself with those modes of thought that are essentially algebraic--for example, handling the as yet unknown, inverting and reversing operations, seeing the general in the particular. Becoming aware of these processes, and in control of them, is what it means to think algebraically. (p. 49)

Appealing though this characterization might seem initially, especially the aspect referring to alg[illegible] thinking as a different mode of thought, I would like to quibble with Love's first two processes: handling the as yet unknown, and inverting and reversing operations. These two processes are included in the procedures involved in solving equations. There is ample empirical evidence to show that students are able to manipulate symbolic expressions and equations with a great deal of control and success, but still not be able to do much else in algebra (e.g., Booth, 1984; Kieran, 1984; Matz, 1979; Wagner, Rachlin, & Jensen, 1984). Love's third process, seeing the general in the particular, suggests an ability that is qualitatively different from the other two. However, even this one does not seem to go far enough. I suggest that, for a meaningful characterization of algebraic thinking, it is not sufficient to see the general in the particular; one must also be able to express it algebraically. Otherwise we might only be describing the ability to generalize and not the ability to think algebraically. Generalization is neither equivalent to algebraic thinking, nor does it even require algebra. For algebraic thinking to be different from generalization, I propose that a necessary component is the use of algebraic symbolism to reason about and to express that generalization.

To make the point more clearly on what I mean by algebraic symbolism and how it can be used to reason about and to express general statements, I refer in some detail to an article by Harper (1987).

The historical development of algebraic symbolism is used by Harper as a theoretical framework for analyzing qualitative differences in student ability to represent generalizations of numerical relations. Harper begins by describing the three evolutionary stages through which the development of algebraic symbolism has passed. The rhetorical stage, which belongs to the period before Diophantus (c. 250 AD), was characterized by the use of ordinary language descriptions for solving particular types of problems and lacked the use of symbols or special signs to represent unknowns. The second stage, referred to by Harper as Diophantine, extended from Diophantus who introduced the use of abbreviations for unknown quantities to the end of the sixteenth century. Harper has pointed out that the concern of algebraists during these centuries was exclusively that of discovering the identity of the letter(s), as opposed to an attempt to express the general. The third stage, referred to by Harper as Vietan, was initiated by Vieta's use of letters to stand for given quantities. At this point it became possible to express general solutions and, in fact, to use algebra as a tool for proving rules governing numerical relations. It is this Vietan stage in the development of algebraic symbolism that forms the basis of what I consider algebraic thinking to be.

In his interviews of 144 secondary school pupils from Years 1 to 6, using questions such as:

> "If you are given the sum and the difference of any two numbers, show that you can always find out what the numbers are."

Harper was able to find evidence of the three types of solutions that can be identified in the history of mathematics. With the rhetorical method, the pupil does not use algebraic symbolism but nevertheless specifies a procedure that is general (e.g., "You divide the sum by 2 then divide the difference by 2; then to get the first number add the sum divided by 2 to the difference divided by 2; to get the second number take the difference divided by 2 away from the sum divided by 2."--Harper, 1987, p. 81). With the Diophantine method, the pupil uses a letter (or letters) to represent an unknown quantity (e.g., $x - y = 2$ and $x + y = 8$, solving for x and y) and states that the method can be applied to any numbers but does not use symbols for a general "given" quantity. With the Vietan method, the pupil uses letters for both unknown and given quantities:

Let nos. = x and y
m = sum of x and y
n = difference of x and y
General equations: $m = x + y$
$n = x - y$
Add together: $m + n = 2x$
. . . Find x and substitute back for y.

It is important to note that the Diophantine solution assumes that the same process can be carried on no matter which sum and difference are chosen and, thus, x is an unknown whose value is to be found. The Vietan solution, on the other hand, has a means of expressing any sum and any difference and of specifying the solution: The two numbers are (m + n)/2 and (m − n)/2. Not only is this solution general, it uses letters rather than conventional numerals to express given quantities.

Only 28 of the 144 pupils of the Harper study used a Vietan type of response to the above problem. Harper points out that the use of this approach rises dramatically in Year 5, but mostly among the more mathematically-able students--20 of the 28 who used a Vietan response were in Year 6. He further notes that these findings accord well with the 8% success rate among Year 4 students of the CSMS study (Küchemann, 1978) on the question: "Which is larger, 2n or n + 2; why?" Thus, it would appear that the use of a Vietan approach is not something that high school students are adept at.

Another example illustrating the Vietan function of letters to express the general is provided by the work of Chevallard and Conne (1984) who have documented students' use of algebraic symbolism as a tool for proving rules governing numerical relations. These two researchers presented an eighth-grade student with a sequence of questions that included the following:

> Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two. ... Now do it with another three consecutive numbers. ... Can you explain it with numbers? ... Can you use algebra to explain it?

The student began with the three consecutive numbers 3, 4, and 5, which led to the calculation of 16 − 15 to yield the result of 1.

He then tried out the numbers 10, 11, and 12, which led to the same result. When asked to explain what was happening, using algebra, he at first tried out $x^2 - yz = 1$--simply replacing all of the "given" numbers by letters. Having then realized that the use of only one letter would be better ("Puis après j'ai pensé qu'on prenait simplement le chiffre au carré qu'on remplaçait par une lettre puis les autres c'est ce chiffre plus un et ce chiffre moins un"--Chevallard & Conne, 1984, p. 8), the student proved the rule governing this numerical relation with the formulation, $x^2 - ((x + 1)(x - 1)) = 1$. Chevallard and Conne point out that this student, though only in the eighth grade, was one who had unusual facility with algebraic representations and their use as thinking tools.

Many other students have, however, been found not to be so successful in using algebraic symbolism as a tool with which to think about and to express general numerical relationships. Lee and Wheeler (1987), in their study of students' conceptions of generalization and justification, tested 354 Grade 10 students on subsets of their questionnaire and then interviewed 25 of these students. One of the questions they presented to the students was the following:

> A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with."
> *Using algebra, show that she is right.*

Of the 118 students who were given this problem, only 9 set up $(5x + 12 - x)/4$ and then algebraically worked it down to $x + 3$. Four of these 9 students then went on to "demonstrate further" by substituting a couple of numerical values for x. Thirty-four others set up $(5x + 12 - x)/4 = 3 + x$ and then proceeded to simplify the left side, yet did not base their conclusions on their algebraic work. They then worked numerical examples and concluded from these examples. The interviews provided further evidence of students' ignoring their algebra.

Another question from the Lee and Wheeler study was the following:

> Show, using algebra, that the sum of two consecutive numbers is always an odd number.

Although the way in which the question was formulated is different from Chevallard and Conne's in that the latter began by asking students to work initially with numerical examples to see what they came up with, it does ask what Chevallard and Conne eventually requested of their subject. Only 7% of Lee and Wheeler's students succeeded on this question. Nevertheless, the interviews showed that students do appreciate an algebraic demonstration when they or someone else produces it, but they are happier with their own numerical examples.

The study by Wheeler and Lee showed that "formulating the algebraic generalisation was not a major problem for the students who chose to do so: using it and appreciating it as a general statement was where these students failed" (Lee & Wheeler, 1987, p. 149). Evidence illustrating that the majority of high school students do not see algebra as a tool for generalization and justification was also seen in the results of the Harper (1987) study. A historical perspective suggests that the "big picture" of present-day algebra involves two major components: (a) the use of algebraic symbolism as a tool to solve specific problems (i.e., discovering the identity of the letter(s)), and (b) the use of algebraic symbolism to express general solutions and as a tool for proving rules governing numerical relations. Research evidence has shown that our schools are better at equipping students to do the former than the latter.

This paper has taken a restricted perspective in that it has focused on algebraic thinking as characterized by the latter component of the "big picture" of algebra. For some authors (e.g., Open University, 1985), the main idea of algebra is that it is a means of representing and manipulating generality and, thus, they see algebraic thinking everywhere, even in the recording of geometric transformations. There are some advantages to taking a more restricted perspective, that is, in not viewing algebraic thinking as equivalent to algebra or to generalization. One of these is that it can provide researchers with an entry point for investigating students' conceptualizations in a well-defined area and subsequently guide them in conducting teaching experiments aimed at helping students develop meaning for this essential aspect of algebra. A second advantage to taking this perspective on algebraic thinking--and it may be the more important one at the present time--is that it can alert us to the fact that computer technology and, consequently, most computer-based approaches to the teaching of algebra are not ideally suited to incorporating this aspect of algebra into their programmes. Algebraic thinking--as characterized in this paper--could well become an area of algebra that is taught even less (if at all) in computer environments than it is now taught in traditional algebra courses. That, in my opinion, would be a real loss.

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References

Booth, L. R. (1984). Algebra: Children's strategies and errors. Windsor, Berkshire: NFER-Nelson.

Chevallard, Y., & Conne, F. (1984). Jalons à propos d'algèbre. Interactions didactiques, 3, 1-54.

Harper, E. (1987). Ghosts of Diophantus. Educational Studies in Mathematics, 18, 75-90.

Kieran, C. (1984). A comparison between novice and more-expert algebra students on tasks dealing with the equivalence of equations. In J. M. Moser (Ed.), Proceedings of the Sixth Annual Meeting of PME-NA (pp. 83-91). Madison: University of Wisconsin.

Küchemann, D. (1978). Children's understanding of numerical variables. Mathematics in School, 7(4), 23-26.

Lee, L., & Wheeler, D. (1987). Algebraic thinking in high school students: Their conceptions of generalisation and justification (Research report). Montreal, Quebec: Concordia University, Mathematics Department.

Love, E. (1986). What is algebra? Mathematics Teaching, 117, 48-50.

Matz, M. (1979). Towards a process model for high school algebra errors (Working paper 181). Cambridge: Massachusetts Institute of Technology, Artificial Intelligence Laboratory.

Open University. (1985). Routes to/roots of algebra. Milton Keynes, UK: The Open University Press, Centre for Mathematics Education.

Wagner, S., & Kieran, C. (Eds.). (1989). Research issues in the learning and teaching of algebra. Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Lawrence Erlbaum.

Wagner, S., Rachlin, S. L., & Jensen, R. J. (1984). Algebra Learning Project: Final report. Athens: University of Georgia, Department of Mathematics Education.

A STRUCTURAL CONCEPTUAL MODEL FOR INVESTIGATING SOME COGNITIVE ASPECTS OF PROBLEM-SOLVING

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A structural conceptual model of programming strategies, together with its empirical verifications, is presented. the paper stresses the need for a complete structural model rather than studying isolated concepts. Guttman's Facet Theory has been employed for the construction and the validation of the proposed model. Three facets define the conceptual model: Knowledge type (content, structure and implementation), Language (level of abstraction), and task Familiarity (level of analogical generalization). The structural lawfulness revealed in the analysis reflects the associative connections among problem solving concepts in the learner's mind. Some cognitive aspects as well as educational implications are discussed.

INTRODUCTION

In order to investigate the relationship between task performing and thinking, there is a need for a conceptual model based on the cognitive analysis of the performance. The construction of a structural conceptual model, and its empirical verification, will be presented in this paper.

The proposed model is based on two assumptions. The first assumption is that conceptual components of a learner's thinking can be inferred from observed behaviors. The second assumption is that connections among those components are reflected through correlations between the corresponding observable behaviors. That is, closely connected components in the model will tend to produce highly associated behaviors. Therefore, a step towards the intended model should be to assess a typology of students' strategies in performing problem-solving tasks. The classifying rules, according to which strategies will be classified, will serve as facets of the structural conceptual model. The term structural refers to the structure of interrelations among those facets.

This paper stresses the need to investigate cognitive aspects of problem solving from a complete structural conceptual model rather than from studying isolated concepts. The representation of all the components in one encompassing structure enables us to reveal relationships which are not evident in studying each component separately. Guttman's Facet-Theory (Guttman, 1957) turns out to be particularly appropriate in pursuing this process of model building.

The first step towards constructing a conceptual model is to define the world of discourse. The universe of content which was defined in this study is concerned with strategies in performing programming tasks. Nevertheless, most of the implications for teaching and curriculum development can easily be transferred to other learning domains or even be generalized to phenomena independent of school subjects.

The paper starts with briefly introducing Guttman's Facet Theory. In the second and the third sections, this methodology is employed for constructing and validating a proposed conceptual model. The last section is a discussion of some cognitive aspects of problem solving and possible educational implications.

FACET THEORY - A METHODOLOGY FOR CONSTRUCTING STRUCTURAL MODELS

According to Guttman's approach (Guttman 1957), a universe of content is defined as a Cartesian product of several facets. A facet according to Guttman's terminology, is a classification rule according to which variables are classified. Each facet is a component set of the Cartesian set which defines the universe of content. The representation of each strategy as a unique element of the Cartesian set, reveals its similarities and differences with any other strategy.

THE METHOD

Choosing Relevant Facets for the Proposed Conceptual Model.

The concept of a model implies that it is a simplification of the real world. Therefore, one has to decide what the aspects of the world to be modelled are. For the sake of simplicity and generality, we decided to concentrate on three facets only. The facets were chosen on the basis of existing theories dealing with human information processing and problem solving (reviewed in: Krumholtz 1987, Bar-On & Krumholtz, forthcoming). The facets and their elements are described in the following sections.

The familiarity facet

In this study tasks were ordered from familiar to unfamiliar, according to the familiarity of the objects dealt with. The objects were the basic geometrical forms (squares and hexagons), which served as basic building blocks. Squares were considered to be more familiar than hexagons.

The language facet

The second facet is concerned with the language in which the description of the solution is expressed. Languages can be ordered according to their level of formality. This hierarchy is consistent with the one proposed by Chomsky for formal generative grammars. The two structs (elements in the facet) chosen in this study were: natural language (Hebrew) and computer language (Logo), which represent two extremes on the language hierarchy.

The knowledge type facet

The third facet is concerned with the type of knowledge which is presumably employed by the student. We propose to distinguish among three types of knowledge:

a. Content related knowledge - refers to domain specific concepts and their meaning. For example, the strategy: "declarative verbal description" (as opposed to a procedural one), which cannot be employed without referring to a specific content (i.e. the geometric figure).

b. Knowledge about structure or organization - is manifested as structuring a description or a computer program, by employing modules, constructing plans and spatial organization. This knowledge is independent of a specific content.

c. Knowledge about implementation schemes - which can be treated independently of a specific content or structure. In our case it was a graphical task and the implementation strategies were classified according to the spatial representation system which they employed: Extrinsic spatial representation system, in this study, the well-known Cartesian rectangular system, and Intrinsic spatial representation system which was adopted by the "turtle language" (Abelson and DiSessa, 1981). The Logo computer language, used in this study, enables the usage of both extrinsic and intrinsic spatial representation systems.

A Definitional System for the Observations

In order to achieve an empirical validation of the proposed model, and to check the correspondence between the predic[illegible] . the observed structure of interrelations, a specific definitional system for the observations was defined. The Cartesian product of the three facets defines twelve ($2 \cdot 2 \cdot 3=12$) possible different types of strategies, which can be employed to define systematically, specific programming strategies (i.e. the items). The observable behaviors were the extent of employing each strategy by each subject (i.e. the observations). Sixty observable behaviors were defined in a specific experimental set-up and thus, could be quantitatively assessed.

THE EXPERIMENT

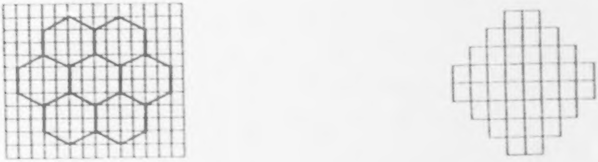
Population and Experimental Set-up

The research population consisted of seventy-eight subjects. All the subjects were novice programmers and those courses were their first experience with computers. The subjects studied according to a curriculum, which was especially designed for this study. The special design intended to emphasize the higher cognitive skills of programming and to expose the subjects to the two spatial representation systems.

The experimental tasks

The experiment consisted of two programming tasks (Figure I), in each a drawing of a geometrical figure was presented. Each of the two tasks consisted of two sub-tasks: description in natural language and programming in computer language. The Logo code was written without using a computer to verify it.

Figure I: The two geometrical figures which have been used in the experimental tasks.



"Squares figure"

"Hexagons figure"

Items and Categories for Observation

As mentioned above, the programming behaviors of the learners were observed and their preferences for the predefined strategies were recorded. The following are the categories for evaluating the items (a detailed description in: Bar-On & Krumholtz, forthcoming).

Items concerning natural language

Item 1. Using declarative verbal description - The evaluation of this item ranges from a procedural description to a declarative one. Characteristics of procedural descriptions are: using verbs (e.g. do, draw, turn), dynamic description of constructing and considering the temporal order of performance (e.g. first, then, at last). Characteristics of declarative descriptions are static, using words like: there are, built of, etc., ignoring the temporal order of construction of the figure.

Item 2. Holist perception of a figure - This item concerns the order of referring to the figure when describing it. Possible descriptions range from first describing the figure as a whole, to starting with a description of the details of the figure.

Item 3. Complexity of "basic building-block" - This item relates to the most inclusive "basic building-block" used for constructing the figure.

Item 4. Ignoring details. **Item 5.** Spatial organization as measured from subject's indication of the location of the figure on the paper.

Items concerning the use of formal language

Item 6. "Top-Down" planning. **Item 7.** Structuring by using sub-procedures.

Item 8. Explicit reference to location.

Item 9. Giving meaningful name to the main procedure.

Item 10. Organization of the produced program.

The last three items (i.e. 11, 12, 13) refer to preference of extrinsic spatial representation system.

Method of Analysis

As mentioned, the analysis of the observations in this study, was concerned with assessing a typology of programming strategies. The methodology of treating this problem is the Non-metric Smallest Space Analysis (SSA). This analysis is performed on the intercorrelation matric. The computer program employs the algorithm suggested by Guttman (1968), for calculating the smallest space (minimal number of dimensions), required to represent the structure of interrelations among the strategies. In order to reveal this structure, these interrelations are represented geometrically. Items are represented as points in an Euclidean space, where the distances between points reflect their dissimilarities. The desired space that enables such an inverse relationship to exist between the observed correlations and the geomatrical distances is one with the minimal number of dimensions. Assessing the empirical verification for the proposed structural model from this analysis, will be explained in the results section.

RESULTS

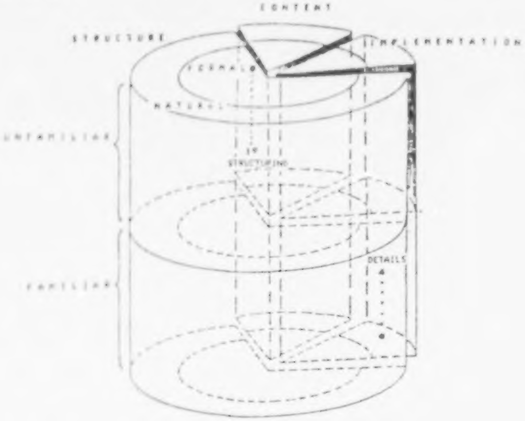
Typology of Programming Strategies - The Cylindrical Lawfulness

Prior to the application of the Smallest Space Analysis (SSA-I), several preparatory steps had been taken, i.e. the selection of items to be analysed and the computation of the intercorrelations among the selected items (detailed analysis in: Bar-On & Krumholtz).

The smallest space for the programming strategies is three dimensional, and has been shown to have a cylindrical lawfulness (figure II), which corresponds to the three proposed facets.

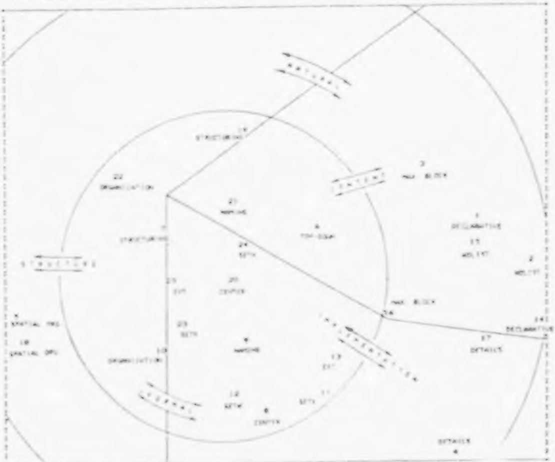
The familiarity of the task facet is represented by a separate dimension which is orthogonal to the language-knowledge plain. The middle horizontal circular disk (in figure II), separates the items concerning the "Squares figure" from those concerning the "Hexagons figure".

Figure II: A schematic CYLINDREX represents the structure of interelations among the strategies when all three facets are considered.


CONTENT
STRUCTURE
IMPLEMENTATION
FORMAL
NATURAL
UNFAMILIAR
19
STRUCTURING
DETAILS
FAMILIAR

The three dimensional cylinder structure can be displayed as two separate orthogonal projections onto two dimensional space:

Figure III: Output of the SSA program: projection of the three dimensional CYLINDREX onto two dimensional space. This figure represents the structure of interrelations among the programming strategies when only the facets of the "language" and "knowledge type" are considered.



As can be seen from figure III, the language and the knowledge-type plain can be divided into two concentric circular bands. The peripheral band contains the points corresponding to strategies which are expressed in natural language, whereas strategies expressed in formal (computer) language, occupy the inner band. Another structural lawfulness revealed in the same plain is the partition to three wedge-like regions emanating from the same common origin, corresponding to the three types of knowledge.

DISCUSSION

The cognitive aspects involved in problem solving which will be discussed are based on the constructed and verified model. The underlying assumption was that the structure of empirical interrelations among observed behaviors, reflects the associative connections among the problem-solving concepts in the learner's mind. Further it was assumed that hypothetical cognitive constructs explain the observable behaviors.

According to that model, the knowledge can be divided into content, structure and implementation types of knowledge. The three types of knowledge can be expressed in various levels of abstraction and formality. The formal expressions of the different knowledge types tend to be highly correlated, while the informal expressions of content, structure and implementation aspects are more distinct.

The structural lawfulness revealed in the language and the knowledge-type plain implies a relation of hierarchy to hold between natural language and formal one. The rationale for this relation is that a formal computer language is conceptualized as a restriction of a natural language. From this result it can be implied that performing of programm[illegible] should always start with using the natural language before applying the formal computer language. This suggestion holds whatever type of knowledge of the three is being discussed.

In order to identify the source of difficulties the learner has in performing any given task, it is necessary to distinguish between two stages in the process of problem solving. The first concerns the understanding of the problem at hand and the ability to describe the solution in natural language. The second stage deals with formulating the problem as a computational process expressed in a formal language. Very often the learner's difficulties are in making the connection between the informal level (using the natural language) and the formal one. It is worthwhile indicating that these two stages can be utilized both in the computer science discipline, where the formal language is any computer language, and in mathematics where the computational process is expressed in the appropriate formal mathematics language.

The structure which represents the relations among elements of the knowledge types, reflects no notion of order among the elements. This implies that no preference among these elements is suggested when a task is to be performed. This finding has an important educational implication in the process of teaching. It means that it is important to analyse a given task concerning each of those three types of knowledge separately. Dealing with each element independently of the two others, emphasizes different aspects of the same problem and thus, enables better understanding of the problem at hand. Those relations among the knowledge types hold whatever language of the two is being used. For the designing of tasks in teaching it means, that students will be asked to relate to the three types of knowledge expressing them in various levels of abstraction and formality.

The structure of relation that was revealed between the first facet on one hand, and the second and the third on the other hand, reflects no dependency. This means, the level of familiarity of the task is independent of the two other facets since it depends on the individual learner. The relation implies as well, that the interrelations between language and type of knowledge will remain unchanged in different tasks. The findings emphasize the importance of exhibiting both familiar and unfamiliar tasks in the teaching processes, while starting with familiar problems. The finding that this structure is preserved across different analogical tasks is consistent with the meaning of analogy as a structure preserving transformation.

REFERENCES

Abelson H. & DiSessa A. (1981): **Turtle Geometry**, MIT Press.

Bar-On E. & Krumholtz N. (forthcoming): **A Structural Conceptual Model of Programming Strategies.**

Guttman L. (1957): Introduction to Facet Design and Analysis. **Proceedings of the 15th Internationl Congress of Psychology**, Brussels, Amsterdam: North-Holland Publishing Company, 130-132.

Guttman L. (1968): A General Nonmetric Technique for Finding the Smallest Coordinate Space for a Configuration of Points. **Psychometrika**, 36, 329-347.

Krumholtz N. (1987): **Typology of Graphical Programming Strategies.** Unpublished doctoral dissertation, Haifa, Technion (in Hebrew).

THE EFFECT OF SETTING AND NUMERICAL CONTENT ON THE DIFFICULTY OF RATIO TASKS

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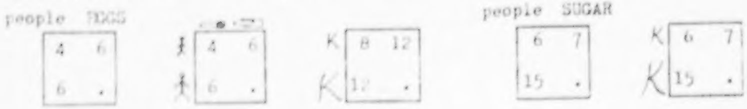
Summary. Data from three written ratio tests confirm findings of other researchers, that Setting and Numerical Content can have a marked effect on the difficulty of ratio tasks. Further, the data throw light on the effect of Setting and Numerical Content on students' preference for Within ratio or Between ratio procedures.

Introduction. Three closely related ratio tests (Tests Rx, Ry, Rz) were developed, each containing about 30 items (a mixture of missing value and comparison tasks) and taking about half an hour to administer. Each test was given to just over 150 secondary school students (156, 153, 154 for Rx, Ry, Rz respectively). Though the samples were different, they were comparable, in as much as the tests were distributed randomly to students within their mathematics classes. The students were in their 2nd, 3rd or 4th year of secondary school (that is, between 13 and 15 years old) and five schools in England and one in Wales took part. No attempt was made to control for age, mathematical attainment or for other background variables and the data are not intended to provide norms; however, they do allow the effect of different item types on students' performance to be compared.

The tests were developed to investigate further some of the findings of Hart(1981), Karplus et al (1983), Vergnaud(1983) and others. Some of the items were based on tasks in the SMP texts (Harper et al, 1987) with which the writer is involved.

Setting. The study of children's understanding of ratio undertaken by Hart(1981) as part of the CSMS study seemed to indicate that students could more readily identify a ratio relationship in, for example, a Setting involving a recipe than in a Setting involving geometric enlargement. It was decided to investigate this further in the present study, by using items related to Hart's but in which the Numerical Content remained the same, or at least similar, while the Setting changed. Data from the resulting items add weight to Hart's findings. Thus for example, in the items shown schematically below, the Recipe item 1Z is much easier than item 1Y, even though they involve identical numbers, which in turn is much easier than the Enlargement item 6.3Y, even though it involves the same "ratio factor" (x1½). Likewise, the Recipe item 2.1X is much easier than the Enlargement item 6.1X though they have the same Numerical Content.

Item 1Z (64% facility) Item 1Y (45% facility) Item 6.3Y (29% facility) Item 2.1X (49% facility) Item 6.1X (22% facility)



Item 1Y is identical to an item used by Hart and is based on the well known Mr Short & Mr Tall task devised by Karplus(1970). Hart found that about half the students in her samples gave the response 8, which is consistent with their using the Addition Strategy (4+2=6, 6+2=8). In the present study, 41% of the students gave this response to item 1Y and almost the same proportion gave the corresponding response of 16 to item 6.3Y. On the other hand, only 19% gave the Addition Strategy response to item 1Z.

It is of interest to speculate why the enlargement Setting, in particular, provokes more Addition Strategy responses than the recipe Setting (in turn, this might throw light on the finding that the recipe Setting is easier). It is possible that students less readily see that the Addition Strategy is inappropriate in the enlargement Setting: increasing the sides of a rectangle, say, by the same amount still produces a rectangle, and one whose shape might be difficult to distinguish from the original if the increase is relatively small; on the other hand, having two more eggs for two more people might well be seen as unjust, given that in the original recipe there are more eggs than people.

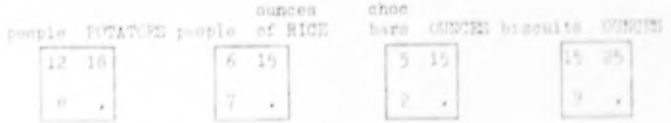
Item 1Z was also given to 31 adult students at the Institute of Education, who were asked to write down the method they used, as well as giving the answer. Just over one-third wrote that they had used a method of this sort:

Half as many people again, so half as many eggs again
(that is, 6 eggs + 3 eggs = 9 eggs or 4,6 + 2,3 = 6,9).

This approach has variously been called Rated Addition (Carraher,1986), Scalar Decomposition (Vergnaud,1983) or a Build Up method (Küchemann,1981; Hart,1981). It seems likely that a sizeable proportion of school students would also use this method on item 1Z. However, the method does not seem well suited to a geometric Setting. While one can produce a recipe for 6 people by combining recipes for 4 people and 2 people, it is not so easy to see how an enlarged geometric shape might be produced by combining an object with one that is half as big (especially if the object is a curly K...). In turn, this suggests that for some students at least, the enlargement Setting is more difficult.

Numerical Content. In general, the effect of Numerical Content on facility seemed predictable. Items with ratio factors of x1½ and x2½ tended to have comparable facilities for a given Setting (eg 9.1Z and 2Y, below); items involving simpler ratio factors (eg x3, as in 5Y below) tended to be substantially easier, whilst items with more complex ratio factors (eg 1⅔, as in 5Z below) were much harder.

Item 9.1Z (64% facility) | Item 2Y (65% facility) | Item 5Y (85% facility) | Item 5Z (30% facility)

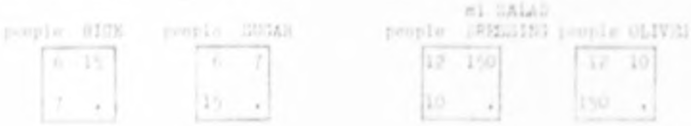


However, Numerical Content seemed to exert a secondary and more subtle effect on facility in that it seemed to influence students' choice of Within ratio or Between ratio procedures. This is discussed next.

Within and Between Ratios. For any ratio item of the sort discussed here, it is possible to construct two kinds of ratio. For example, for the Rice Salad item (item 2Y) it is possible to construct the ratio of number of people and the amount of rice (6:15) or the ratio of the different numbers of people (or the different amounts of rice) (6:7). The first kind is commonly called a Within or Function ratio, the second a Between or Scalar ratio (Karplus et al, 1983; Vergnaud, 1983).

For the items below, which are all in a recipe Setting, it turns out that the easier item in each pair is the one where the Within ratio is simpler that the Between ratio (I would argue that 6:15 or x2½ is simpler than 6:7 or x1⅙; and that 12:150 or x12½ is simpler than 12:10 or x⅚).

Item 2Y (x2½ →) (65% facility) | Item 7.1X (x2½ ↓) (49% facility) | Item 9.4Z (x12½ →) (40% facility) | Item 7.4X (x12½ ↓) (35% facility)



The above suggests that, in a recipe Setting at least, students generally prefer to transform the numbers that form the Within ratio rather than the Between ratio. However, this does not necessarily mean that they prefer Within ratio procedures. Consider the numbers 6,15 that form the Within ratio pair for item 2Y.

It is possible to transform the numbers using a Within or Between ratio procedure, as is illustrated in the diagram below. The Within ratio procedure might involve identifying the operator x$2\frac{1}{2}$ ($6\times 2\frac{1}{2}=15$) and applying it to 7 ($7\times 2\frac{1}{2}=17\frac{1}{2}$, so 7 people require $17\frac{1}{2}$ ounces of rice). On the other hand, the Within ratio pair 6,15 can be transformed into the pair 3,$7\frac{1}{2}$, say, and in particular, by using the Unitary method, into the pair 1,$2\frac{1}{2}$. Though the latter might involve the same arithmetic as is used to find the operator x$2\frac{1}{2}$ ($15\div 6$, say), Vergnaud(1983) makes the very important point that in this case a Between ratio procedure is being used: 6 people is transformed into 1 person, 15 ounces of rice into $2\frac{1}{2}$ ounces; there is no transformation from number of people to number of ounces of rice.

A Within ratio procedure on
item 2Y using the operator x $2\frac{1}{2}$

people RICE

$$\begin{tikzcd}
6 \arrow[r, "\times 2\frac{1}{2}"'] & 15 \\
7 \arrow[r] & .
\end{tikzcd}$$

A Between ratio procedure on
item 2Y using the Unitary method

people RICE

$$\begin{tikzcd}
6 \arrow[d, "\div 6"] & 15 \arrow[d] \\
1 \arrow[d, "\times 7"] & 2\frac{1}{2} \arrow[d] \\
7 & .
\end{tikzcd}$$

To investigate students' choice of procedure further, the test sheets on which students had written their responses to items 2Y and 2.1X were scrutinised for any indications of the methods they might have used.

Most students did not show any working, and those who did tended to provide working that was ambiguous, as in the example below. The working is for item 2Y and leads to the correct response, but it is not clear whether the Unitary method is being used ($2\frac{1}{2}$ ounces per person, etc) or whether $2\frac{1}{2}$ is being used as an operator ($2\frac{1}{2}$ times as many ounces as people, etc).

$15 \div 6 = 2.5 \quad 2.5 \times 7 = 17.5$

Overall, of the 94 students who answered 2Y correctly, 39 showed some working, of whom 37 wrote working that included the intermediate value $2\frac{1}{2}$. (The remaining two students wrote down the expression $15\div 6\times 7$.) This strongly suggests that most of the students who gave a correct response to item 2Y either used a Within ratio procedure involving an operator (x$2\frac{1}{2}$) or used a Between ratio procedure involving the Unitary method (and/or Rated Addition).

Interestingly, none of the students who answered 2Y correctly, wrote down the intermediate value $1\frac{1}{6}$(or 1.16, etc), which would almost certainly have indicated a Between ratio procedure with $1\frac{1}{6}$ as operator. This contrasts quite strongly with item 2.1X, which involves the same numbers as 2Y, but transposed. Here, 69 students answered the item correctly, with 28 students showing some working. Of these 28, 12 gave the intermediate value $1\frac{1}{6}$, whilst only 4 gave the intermediate value $2\frac{1}{2}$. The relative frequencies of the values $1\frac{1}{6}$ and $2\frac{1}{2}$ suggest that, as with item 2Y, more students chose to transform the Within ratio pair (using either a Within ratio procedure involving an operator, or a Between ratio procedure involving the Unitary method) -despite the fact that this time the Within ratio pair (6,7) is arithmetically more complex that the Between ratio pair (6,15).

A similar picture emerges from an examination of students' working for item 9Y, shown here on the right. For this item the ratio pairs 21,35 and 21,30 would appear to be of roughly equal complexity, but of the 20 students who answered the item correctly and showed working, 16 gave the intermediate value derived from evaluating 35÷21, but none gave the value that corresponds to 30÷21.

Item 9Y
(32% facility)

ticks	SECONDS
21	35
30	.

The scrutiny of students' scripts confirms that in some settings (in particular those involving recipes), students prefer to transform the Within ratio pair. However, it is still not clear whether they prefer to do this using Within ratio or Between ratio procedures. To pursue this further, some adult students at the Institute of Education were given a recipe item (item 1Z) and asked to write down as clearly as possible the method they used to solve it. Thirty-one students took part, of whom 9 used a Between ratio Unitary Method and a further 11 used a Between ratio procedure involving Rated Addition. Three students clearly used a Between ratio procedure using an operator (eg 6 people = 3/2 x 4 people, so no. eggs = 3/2 x 6 eggs). Two students seemed to use the Rule of Three, and the remaining 6 students' explanations were ambiguous. No one unambiguously used a Within ratio method.

This small supplementary study, then, strongly suggests that, for recipe settings, students not only prefer Within ratio pairs, but prefer to transform them using Between ratio procedures (in particular, involving the Unitary method or Rated Addition). Further credence is given to this by a consideration of the meanings that might be attached to the elements in these procedures: while the Unitary method and Rated Addition are always working with "states" or "entities" (4 people need 6 eggs, so 2 people need 3 eggs, 1 person needs 1½ eggs, etc), the within ratio operator procedure seems to be working with something less tangible, namely a relationship between states, which might also seem rather contorted (whatever the number of people, there are 1½ times as many eggs...).

Somewhat different conclusions have been drawn from studies by Karplus et al (1983) and Vergnaud(1983). Karplus found no clear preference for Within or Between ratio procedures. However, this conclusion is based, essentially, on data from just one item (shown below, left) which, furthermore, can be solved directly by using a whole number operator (x2 or x5). Because of this, the item would seem to "discourage" the Unitary method; at the same time, no mention is made of this method and it is possible that any occurrences of it would have been counted as instances of a within ratio procedure as it involves transforming the Within ratio pair 3,9.

Vergnaud did take account of the Unitary method. He found that even with an item like the one below (right), which would seem to favour transformation of the Within ratio pair, more students used Between ratio procedures than Within ratio procedures, but the Between ratio procedure was generally not the Unitary method. However, as with the Karplus item, Vergnaud's items can be solved in one step, using a whole number operator, which might supress use of the Unitary method. It can be argued, therefore, that the present study extends rather than contradicts that of Vergnaud.

Karplus Ratio Item

	laps	TIME
Jane	3	9
Phyllis	6	15

Vergnaud Ratio Item

hours	LITRES of oil
7	21
84	.

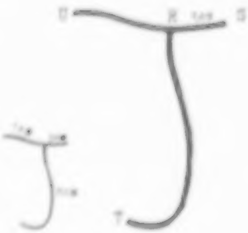
In comparing the Karplus and Vergnaud studies with the present study, it should be noted that neither Karplus's nor Vergnaud's items used recipe Settings, though they did use Settings involving quantities of different dimensions (laps and minutes; hours and litres). The Enlargement items in the present study involve quantities of exactly the same kind (cm lengths) and it is interesting to observe that for these items the preference for transforming Within ratio pairs rather than Between ratio pairs seems to be reversed (though not necessarily the preference for Between ratio procedures). Thus, for example, for the two items shown below (in full and schematically), 6.1Z is answered no more successfully than 6.2Z, even though the ratio pair 2,8 is arithmetically much simpler than 2,5. This finding needs to be investigated further.

Items 6.1Z and 6.2Z

These two Js are exactly the same shape.

How long is the curve RT? ....

[illegible] ....



Item 6.1Z
(50% facility)

J	2	8
J	5	.

Item 6.2Z
(50% facility)

J	2	5
J	5	.

To summarise, the study supports the findings of other researchers that Numerical Content and Setting can have a marked effect on students' success rates on ratio tasks. In particular, the study showns that students are more likely to use effective strategies on tasks in a recipe Setting than ones that involve geometric enlargement. The study also suggests that in tasks that involve a recipe Setting, students prefer to transform Within rather than Between ratio pairs, but that they prefer to do this using Between ratio procedures such as the Unitary method. For tasks in a geometric Setting, the first of these preferences seems to be reversed.

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References

CARRAHER, T (1986): Rated addition: a correct additive solution for proportion problems, *Proceedings of the Tenth International Conference of Psychology of Mathematics Education*, University of London.

HARPER et al (1987): *SMP: Mathematics for Secondary Schools, Book 2 Red Track* and *Book 2 Blue Track*. Longman, Harlow.

HART, K (1981): Ratio, in HART, K (ed), *Children's Understanding of Mathematics, 11-16*. Murray, London.

KARPLUS, R & PETERSON, R (1970): Intellectual development beyond elementary school II: ratio, a survey, *School Science and Mathematics*, *LXX*, 9, Dec 1970.

KARPLUS, R, PULOS, S & STAGE, E (1983): Early adolescents' proportional reasoning on "rate" problems, *Educational Studies in Mathematics*, *14*, 219-233.

KÜCHEMANN, D (1981): Cognitive demand of secondary school mathematics items, *Educational Studies in Mathematics*, 12, 301-316.

VERGNAUD, G (1983): Multiplicative structures, in LESH, R & LANDAU, M (eds), *Acquisition of Mathematics Concepts and Processes*. Academic Press, London.

method and it is possible that any occurrences of it would have been counted as instances of a within ratio procedure as it involves transforming the Within ratio pair 1,3.

Vergnaud did take account of the Unitary method. He found that even with an item like the one below (right), which would seem to favour transformation of the Within ratio pair, more students used Between ratio procedures than Within ratio procedures, but the Between ratio procedure was generally not the Unitary method. However, as with the Karplus item, Vergnaud's items can be solved in one step, using a whole number operator, which might supress use of the Unitary method. It can be argued, therefore, that the present study extends rather than contradicts that of Vergnaud.

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hours	LITRES of oil
7	21
54	.

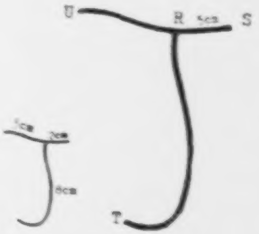
In comparing the Karplus and Vergnaud studies with the present study, it should be noted that neither Karplus's nor Vergnaud's items used recipe Settings, though they did use Settings involving quantities of different dimensions (laps and minutes; hours and litres). The Enlargement items in the present study involve quantities of exactly the same kind (cm lengths) and it is interesting to observe that for these items the preference for transforming Within ratio pairs rather than Between ratio pairs seems to be reversed (though not necessarily the preference for Between ratio procedures). Thus, for example, for the two items shown below (in full and schematically), 6.1Z is answered no more successfully than 6.2Z, even though the ratio pair 2,8 is arithmetically much simpler than 2,5. This finding needs to be investigated further.

Items 6.1Z and 6.2Z

These two Js are exactly
the same shape.

How long
is the curve RT? ....

RU? ....



Item 6.1Z
(30% facility)

J	2	8
J	5	.

Item 6.2Z
(30% facility)

J	2	5
J	5	.

To summarise, the study supports the findings of other researchers that Numerical Content and Setting can have a marked effect on students' success rates on ratio tasks. In particular, the study showns that students are more likely to use effective strategies on tasks in a recipe Setting than ones that involve geometric enlargement. The study also suggests that in tasks that involve a recipe Setting, students prefer to transform Within rather than Between ratio pairs, but that they prefer to do this using Between ratio procedures such as the Unitary method. For tasks in a geometric Setting, the first of these preferences seems to be reversed.

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References

CARRAHER, T (1986): Rated addition: a correct additive solution for proportion problems, Proceedings of the Tenth International Conference of Psychology of Mathematics Education, University of London.

HARPER et al (1987): SMP: Mathematics for Secondary Schools, Book 2 Red Track and Book 2 Blue Track. Longman, Harlow.

HART, K (1981): Ratio, in HART, K (ed), Children's Understanding of Mathematics, 11-16. Murray, London.

KARPLUS, R & PETERSON, R (1970): Intellectual development beyond elementary school II: ratio, a survey, School Science and Mathematics, LXX, 9, Dec 1970.

KARPLUS, R, PULOS, S & STAGE, E (1983): Early adolescents' proportional reasoning on "rate" problems, Educational Studies in Mathematics, 14, 219-233.

KÜCHEMANN, D (1981): Cognitive demand of secondary school mathematics items, Educational Studies in Mathematics, 12, 301-316.

VERGNAUD, G (1983): Multiplicative structures, in LESH, R & LANDAU, M (eds), Acquisition of Mathematics Concepts and Processes. Academic Press, London.

SATISFACTION AND REGRET ABOUT THE CHOICE OF MATH

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ABSTRACT

This paper deals with the choice of math as an examination subject and the satisfaction with or regret of this choice. More boys than girls appeared to choose math. We observed a tendency for girls to regret their choice more than boys. Students not choosing math regretted their choice more than students choosing math. This was especially the case for boys at lower difficulty levels. Probably they regretted their lesser future possibilities without math, the main reason for regret of no-math choice. The main reason for regret of the math choice had to do with poor achievement in math. As predicted by the attitude-model of Fishbein & Ajzen (1975), we observed a quite strong relationship between the intended and actual choice of math. We failed to predict satisfaction with the choice at a satisfactory level using decisional variables measured a year ago. However, the role of the careers master's believed opinion was remarkable in this prediction.

INTRODUCTION

Dutch general formative secondary education consists of a Low Level (LL), a Medium Level (ML), and a High Level (HL) of difficulty (in dutch: MAVO, HAVO and VWO, respectively). After passing three years of education LL and ML students have to choose six examination subjects; one type of math may be chosen. The final examination takes place after one or two more years of education for LL and ML students, respectively. HL students have to choose seven examination subjects after passing four years of education, and the final examination takes place after two more years. Two types of math may be chosen: math A (mainly 'applied': HL-A) or math B (mainly 'pure': HL-B). It is also possible to choose neither one or both.

In a longitudinal study various aspects concerning the *intended* choice of math, the *actual* choice of math, and *satisfaction* with the actual choice were studied at all difficulty levels. One aspect concerned the prediction of the intended choice of math by means of the attitude-model of Fishbein & Ajzen (1975), which provides insight into the decision processes underlying a choice. The attitude-model distinguishes two components that influence the intention to perform a behavior: the aggregated *attitude* towards the behavior and the aggregated *subjective norm* about the behavior. The elements of the attitude and the subjective norm are product-terms of probability ratings and importance ratings (for details, see Otten & Kuyper, 1988). Using multiple regression these product-terms (referred to as the *decisional* variables hereafter), together with three other variables, predicted 64%, 76%, 50% and 73% of the variance in the intended choice at LL, ML, HL-A and HL-B, respectively (Kuyper & Meulenbeld, 1988, Otten & Kuyper, 1988).

The other three variables were sex of student, math-achievement (i.e., the mean of the math grades on the previous two reports), and math-requirement (i.e., whether or not math was required for the favored vocational training). The latter variable was a main predictor of intended math choice at all difficulty levels, especially for boys. Notwithstanding the fact that at all levels sex of the student was not included in the regression equation, the differences between the boys' and girls' regression equations could be attributed to gender differences in favored vocational trainings (Otten & Kuyper, 1988).

The purpose of the present paper is to describe the actual choice behavior of the students and the satisfaction with or regret of that choice at all difficulty levels. Do differences exist between type of choice (math vs. no-math) and sex on these variables? Which are (the) major reasons for regret? Given the fact that Fishbein & Ajzen (1975) postulate that intended choice is the main determinant of actual choice behavior, we will examine the relationship between intended and actual choice of math. Finally, we try to predict actual choice and satisfaction with that choice using as predictors the decisional variables, which predicted the intended choice fairly well.

METHOD

The conducted longitudinal study consisted of two measurements: the first *before* the choice of examination subjects (May and June, 1986), the second *after* this choice (May and June, 1987). The results reported in the present paper mainly concern the second measurement. In this measurement 2445 students from 16 secondary schools participated (30% LL; 38% ML; 32% HL). A minority of the students also participated in the first measurement (33% LL; 26% ML; 36% HL).

Variables of interest in the second measurement were: (1) the actual choice of math (0 = *no-math* chosen, 1 = *math* chosen), (2) the satisfaction with the particular choice (1 = very satisfied, 2 = satisfied, 3 = not satisfied but also no regret, 4 = a bit of regret, 5 = regret, 6 = much regret). Students expressing (some) regret of their choice (categories 4, 5 and 6) were offered eleven or eight reasons for this regret, depending on the particular choice regretted (math vs. no-math, respectively). The students were asked to indicate whether the offered reason was a cause of their regret (0 = no cause, ..., 3 = much the cause). At HL these variables were asked for both types of math (A and B).

Only for students participating in both measurements it was possible to relate intended choice (choose math: 1 = not, 2 = maybe not, 3 = maybe not, maybe yes, 4 = maybe yes, 5 = yes) to actual choice behavior. Also only for these students we could predict actual choice behavior and satisfaction with their choice using as predictors the decisional variables measured in the first measurement. These decisional variables consisted of twelve *attitude*-elements and six *subjective norm* elements. Besides these eighteen decisional variables, the variables *sex* of student, *math-achievement* and *math-requirement* were used as predictors in the multiple regression analyses (see also Introduction; Otten & Kuyper, 1988).

RESULTS

Actual Choice

The results regarding actual choice and satisfaction with this choice are summarized in Table 1. The second column of Table 1 shows that more boys than girls chose math as an examination subject. In general the difference is about 30%, except at HL-A ('applied') where it is only 8%.

Satisfaction with the choice

Before discussing the data concerning satisfaction with the choice, represented in Table 1, we want to stress three points relevant for the interpretation of these data. First, it appeared that 7% of the data on the satisfaction variable were missing. This percentage is higher for boys than girls (8% vs. 5%) and higher for no-math choice than for math choice (10% vs. 4%). Speculating on this last finding, it is conceivable that not answering this question is an indication of regret.

Table 1: Total number of boys and girls, percentages of students choosing math, and percentages of satisfaction with and regret of the choice. Missing data are excluded.

		N	% math choice	MATH CHOICE			NO-MATH CHOICE		
				satisf.	middle	regret	satisf.	middle	regret
LL	boys	297	83	81	15	4	48	45	8
	girls	441	51	62	21	17	74	16	10
HL	boys	402	76	74	20	5	45	41	13
	girls	527	48	74	17	9	68	21	10
HL-A	boys	405	67	81	14	4	62	23	15
	girls	373	59	79	16	6	60	24	16
HL-B	boys	405	60	83	13	4	84	11	5
	girls	373	31	76	17	7	80	14	6

The second point relates to the interpretation of the 'not satisfied but also no regret' category. It appeared that a considerable number of students in this category also scored the subsequent reasons for regret, although they were *not* asked to do so. For this reason the category might be better interpreted as 'satisfaction on the one hand, regret on the other'. This category is referred to as the *Middle* category (M) hereafter.

Third, it appeared that the 'regret' and 'much regret' categories were hardly used;

of the relatively few students indicating regret about 75% scored the 'a bit of regret' category. For this reason the three regret categories (categories 4, 5 and 6) are taken together and referred to as *Regret* category (R) hereafter. Also the two satisfaction categories (categories 1 and 2, in which the frequencies were more equal distributed) are taken together and reffered to as the *Satisfaction* category (S).

Table 1, columns 3 through 8, shows that most students (74% S; 18% M; 8% R) were satisfied with their choice. In general it seems that less students were satisfied with a no-math choice (69% S; 21% M; 10% R) than with a math choice (76% S; 17% M; 7% R). The main exception are the LL-girls. Looking at sex, there seems to be a slight difference in satisfaction; girls (72% S; 18% M; 10% R) seem to regret their choice more than boys (75% S; 19% M; 6% R). The exceptions are found in the no-math choice at LL and ML. Here we find a remarkable high percentage of boys with "mixed feelings" about their choice (LL: 46% S, *45% M*, 8% R; ML: 45% S, *41% M*, 13% R).

Considering level of difficulty, Table 1 shows that the most satisfied students are found at HL-B (81% S; 14% M; 5% R), followed by HL-A (73% S; 18% M; 9% R), LL (71% S; 19% M; 10% R) and ML (69% S; 22% M; 9% R). In general, the students were most satisfied with their choice at HL-B (math and no-math choice) and HL-A (math choice only). The students were least satisfied with their no-math choice at HL-A, ML and LL.

Reasons for regret

Only the students indicating regret (Table 1, columns 5 and 8) are included in the analysis of the importance of the reasons for regret. In case of regretting the math choice, eleven reasons were offered. The most important reason was 'math was very difficult this year' (overall mean 2.1), followed by 'afraid of failing the year' (1.8) and 'too low grades on math' (1.6), which are related to the first (and to each other). The next reason was in fact a reformulation of regret 'I wish I had chosen another subject' (1.1), followed by 'I had to spend a lot of time on math at home' (1.1) and 'I did not like math this year' (1.0). The next two reasons state that math is not necessary for the intended (vocational) study and profession (1.0 and 0.9). The least important reasons were 'I did not like the math teacher' (0.7), 'I did not like the fellow students' (0.2), and 'my parents forced me to take math' (0.1).

In case of regretting the no-math choice eight reasons were offered. Evidently the most important reason was 'I have less possibilities without math' (1.9), followed by 'math is required for the (vocational) study I intend to follow' (0.9), 'math is necessary for my future profession' (0.7) - which are related to the first - and 'I wish I had not chosen math' (a reformulation of regret). The other four reasons were not important at all. On both sets of reasons the means for LL, ML, HL-A and HL-B did not deviate systematically from the overall means, neither did the means for boys and girls.

Intended and actual choice

At all difficulty levels the Pearson product-moment correlation between intended and actual choice is quite high: LL $r=0.81$, ML $r=0.93$, HL-A $r=0.79$, and HL-B $r=0.82$

Prediction of actual choice and satisfaction

Table 2 shows the explained variance in various criteria at all difficulty levels by the decisional variables, sex of student, math-achievement and math-requirement. The numbers in column 1 of Table 2 slightly differ from the explained variances mentioned in the Introduction, because the analyses in Table 2 are performed on the students participating in *both* measurements, whereas the before-mentioned explained variances resulted from analyses on all students in measurement 1. Comparing column 1 and 2 in Table 2 reveals that the predictor variables of measurement 1 explain about 20% less variance in the actual choice behavior than in the intended choice behavior. The exception is at ML, where the difference is only 8%. The predictor variables entering the regression equations of actual choice are similar to the equations of intended choice and the latter are reported elsewhere (Otten & Kuyper, 1988; Kuyper & Meulenbeld, 1988).

Table 2: The explained variance ($100 \times R^2$) in intended choice, actual choice, and satisfaction with the particular choice. Between parentheses the maximum number of subjects in the multiple regression analysis.

	INTENDED CHOICE		ACTUAL CHOICE		SATIS. MATH		SATIS. NO-MATH	
LL	63	(195)	43	(195)	27	(156)	29	(33)
ML	75	(201)	67	(201)	5	(116)	12	(83)
HL-A	44	(173)	27	(173)	10	(34)	24	(39)
HL-B	74	(166)	54	(166)	20	(72)	-	(72)

The percentages explained variance in columns 3 and 4 indicate that predicting satisfaction with the actual choice in measurement 2 by the predictor variables of measurement 1 was not quite a success. The best results are found at LL where 29% and 27% of the variance in satisfaction with the math and no-math choice, respectively, was explained. The worst result is found at HL-B where no prediction of the satisfaction with the no-math choice was possible, because of the small correlations between the predictor variables and the criterium. Despite the small amount of explained variance in the satisfaction with the choice, we will briefly discuss the predictor variables entering the regression equations.

In Table 3 the criterium variable, satisfaction with choice, is mirrored; so the higher the score on satisfaction, the more satisfied with the choice. At LL the students were more satisfied with their math choice when they were a boy, had higher math grades on their previous two reports, saw more possibilities in the future with math as an examination subject, spent less time at math home-work and when they liked their math teachers. Considering the no-math choice, LL-students were more satisfied when they saw less future possibilities with math as an examination subject and (peculiarly) expected their math grade to be high at the examination. At ML the students were more satisfied with their math choice when they thought their friend's opinion was *not* to choose math as an examination subject. The ML-students were more satisfied with a no-math choice when math was not a requirement for their favored vocational training and when they expected to stay together with their friend in the same school-class doing so. At HL-A the students were more satisfied with their math choice when they need not to take extra math-lessons and thougt their careers master's opinion was to do so. Also HL-A students were more satisfied with their no-math choice when they thougt their careers master's opinion was to do so and their friend's opinion was, in contrast, to choose it as an subject. Finally, the HL-B students were more satisfied with their math choice when they had higher math grades on their previous two reports and thought math was necessary for their future profession.

Table 3: The elements included in the regression equations predicting the satisfaction with the particular math choice. Represented are the β-weights, the correlation of the predictor with the criterium (x 100) between parentheses; the criterium (satisfaction) is mirrored.

	LL		ML		HL-A		HL-B
	MATH	NO-MATH	MATH	NO-MATH	MATH	NO-MATH	MATH
R^2 x 100	27	29	5	12	10	24	20
REQUIRE				-27 (-24)			
SEX	19 (25)						
ACHIEV	16 (24)						31 (38)
PROFESSION							25 (34)
FUTURE POSSIBLE	30 (37)	-41 (-40)					
HOME WORK	15 (17)						
EXTRA LESSONS					21 (20)		
EXAMINATION MARKS		36 (35)					
MATH TEACHERS	19 (26)						
FRIEND IN CLASS				-25 (-21)			
FRIEND'S OPINION			-21 (-21)			36 (10)	
CAREERS MASTER'S O.					25 (25)	-54 (-37)	

Generally speaking a quite divergent picture emerges in Table 3, regarding the predictors of the satisfaction with the (no-)math choice next year. We like to draw the attention to four elements in this picture. First, the fact that sex of student is a predictor at LL for satisfaction with math choice. Second, both at LL and HL-B the math grades on the previous two reports are predictors of the satisfaction with the math choice. Third, At LL the amount of possibilities in the future with math as an examination subject is a predictor of the satisfaction of both choices. Fourth, at HL-A the careers master' believed opinion about choosing math as an examination subject is also a predictor of the satisfaction with both choices.

DISCUSSION

Despite the efforts of the Dutch government to stimulate girls to choose math as an examination subject, they still choose it less than boys. Only the more 'applied' math version at HL is chosen by (almost) equal numbers of both sexes. Considering the satisfaction with their choice there seems to be some difference between the sexes. There is a slight tendency that girls *really* regret their choice more than boys.

More striking is the fact that not choosing math is more regretted than coosing math. Especially the high percentage of boys at LL and ML having mixed feelings about their no-math choice is surpising. Probably they realize after their choice that they have less possibilities without math, which was the main reason for regret of the no-math choice. On the other hand those 'mixed' boys may not have been the most brilliant math students, considering the fact that the main reasons for regret of the math choice were: 'math was very difficult', 'afraid of failing this year' and 'too low grades on math'.

The relationship between intended choice and actual choice was quite strong, supporting the postulation of the Fishbein & Ajzen model (1975) that intended choice is a determinant of actual choice behavior. Using the decisional variables, sex of student, math-achievement and math-requirement of the first measurement to predict the actual choice in measurement 2 proved to be less successful than the prediction of intended choice. The best prediction was at ML where 67% of the variance in actual choice could be explained, the worst prediction was at HL-A where only 27% of the variance could be explained.

The attempt to predict the satisfaction with the choice with predictor variables measured one year before failed to reach a satisfactory level. The explained variances varied between 0% and 29%. Remarkable was the role of the careers master's believed opinion in predicting satisfaction with the choice at HL-A; irrespective of which opinion, when the students acted accordingly, they were more satisfied with their choice. Also expected possibilities in the future with(out) math and math-achievement appeared to be predictors of satisfaction. The latter two relating to the main reason(s) for regret of the (no-)math choice.

REFERENCES

Fishbein, M. & Ajzen, I. (1975). *Belief, attitude, intention and behavior: An introduction to theory and research.* Reading, Mass.: Addison-Wesley.

Kuyper, H. & Meulenbeld, J. (1988). *De invloed van het gedrag van docenten op de prestaties in, keuze van en attituden ten opzichte van wiskunde van meisjes in het AVO/VWO (The influence of teachers' behavior on the achievements in, choice of and attitudes about mathematics of girls at formative secondary schools).* Report no. 1 and 2. Groningen: RION, Institute of Educational Research.

Otten, W. & Kuyper, H. (1988). Gender and mathematics: The prediction of choice and achievement. In: A. Borbás (Ed.), *The proceedings of the twelfth annual conference of the international group for the Psychology of Mathematics Education, Volume 2.* Veszprém (Hungary): OOK.

INTRINSIC VERSUS EUCLIDEAN GEOMETRY: IS THE DISTINCTION IMPORTANT TO CHILDREN LEARNING WITH THE TURTLE?

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Abstract. This is a report of case - study research whose aim was to investigate children's criteria for choosing between intrinsic and Euclidean geometrical notions while employing the turtle metaphor in a Circle microworld in Logo. Two 11 - 12 year old Logo - experienced children worked collaboratively within the microworld for 24 hours in total. The analysis shows that the children did not seem to find qualitative differences between the nature of intrinsic and Euclidean notions. Instead, their choices were influenced by certain broader aspects of the mathematical situations generated during the study.

Research into the learning processes of children engaged in Logo programming activities has provided substantial evidence that Logo can be used as a means to generate rich mathematical environments for children to act upon in a personally meaningful way (Noss, 1985, Hoyles and Sutherland, in press). However, it does not necessarily follow that children always use the mathematics embedded in the Logo language. Regarding the learning of geometrical content, for instance, researchers have shown that children do not necessarily use geometrical ideas when doing Turtle geometry, an important part of Logo (Hillel and Kieran, 1987, Leron, 1983); instead, they often restrict themselves to the use of perceptual cues in deciding how to change the turtle's state on the screen (Hillel et al, 1986). Furthermore, little attention has been given to the nature of the geometrical content children can actually learn while using the turtle metaphor, i.e. when they identify with the turtle to drive it on the screen.

Analyses of the geometrical structure of Turtle geometry have characterised it as intrinsic (Papert, 1980, Abelson and diSessa, 1981, Harvey, 1985), i.e. as embedding the two following main geometrical principles: a) the turtle's state is uniquely determined by its immediately previous state and b) there is no reference to any part of space outside the turtle's immediate vicinity. Furthermore, Papert and Lawler argue that the use of the turtle metaphor (termed "intrinsic schema") by children draws upon intuitive ideas originating from very early experiences of bodily motion (Papert, 1980, Lawler, 1985).

The research reported here was part of a wider research project aimed at investigating the potential for children to use the turtle metaphor in Logo in order to develop understandings of a wide span of geometrical ideas belonging not only to intrinsic, but also to Euclidean and Coordinate geometry (Kynigos, 1989). The evidence from this research project corroborates the previous view regarding the intuitive nature of the intrinsic schema (Kynigos, in press). The particular case - study reported here involved a pair of children working with a Circle microworld and was aimed at questioning the implicit general assumption that children's intrinsic schema can be a powerful tool mainly in the learning of intrinsic geometry.

OBJECTIVES

The general aim of the Circle microworld case - study was to investigate the criteria children develop in choosing between intrinsic and Euclidean ideas within the context of Turtle geometry.

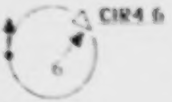
The study involved a pair of children and consisted of two phases. In phase 1 (which was of a preliminary nature), the children participated in a learning sequence involving the construction and use of four circle procedures, each of which embedded specific intrinsic and/or Euclidean notions. The procedures the children wrote as a result of the learning sequence are given in figure 1. In phase 2 (the main research phase), these four circle procedures were treated as primitives of a Circle microworld. The children were given structured tasks involving the construction of figures consisting of compositions of circles. They had the choice of which of the four circle procedures to use in constructing the figures (examples of the task figures are given in figure 2). The specific research objectives in phase 2 were to investigate:

a) the extent to which the children used the geometrical notions embedded in the structured tasks and the nature of the notions they used for constructing the tasks' figures,

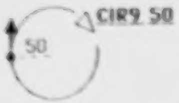
b) the nature of the children's implicit or explicit criteria for choosing intrinsic or Euclidean geometrical notions in their constructions.

METHOD

The pair of 12 year old children participating in the Circle microworld case - study (Alexandros and Valentini), had previously had around 50 hours of experience with Turtle geometry in an informal classroom environment. During the case - study, the children worked collaboratively with the computer for 24 hours in total. The collected data included hard copies of everything that was said, typed and written on paper. Verbatim transcriptions from audio tape, dribble files and written notes were used respectively.

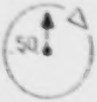


```
TO CIR4 :S
REPEAT 36 [FD :S RT 10]
END
```



```
TO CIR9 :R
RT 5
REPEAT 36 [FD :R * 2 * 3.14 / 36 RT 10]
LT 5
END
```

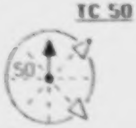
```
TO CIR17 :S
LT 90
PU
FD :S
RT 90
PD
END
```



```
TO CIR19 :S
CIR17 :S
CIR9 :S
CIR18 :S
END
```

```
TO CIR18 :S
RT 90
PU
FD :S
LT 90
PD
END
```

```
TO MOVE :S
PU
FD :S
PD
FD :S / :S
PU
BK :S + :S / :S
PD
END
```



```
TO TC :S
REPEAT 360 [MOVE :S RT 1]
END
```

◁ Denotes a turtle state during the execution of the procedure

⭡ Denotes the turtle's state of transparency

Figure 1 The children's constructions of the Circle microworld's primitives

During the main phase (phase 2) the children were each given a copy of a task figure and asked first of all to individually write down their strategy for solving the task. They were then asked to collaborate by discussing their strategies and trying them out on the computer in the process of changing them or refining them. Finally, the researcher carried out semi - structured interviews at the end of each task in order to gain further insight into their criteria for choosing intrinsic or Euclidean notions. A 90 minute session was allowed for each task. The results from the analysis of the data are outlined below. Specific episodes from the research are described in order to illustrate the main points of the analysis.

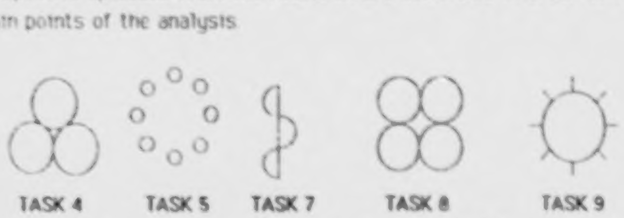


Figure 2 Examples of the structured tasks in phase 2 of the case - study

RESULTS

At the end of the learning sequence in phase 1, even though the children had ample time to construct and use the four circle procedures meaningfully and although they seemed aware of their functioning, there were limits to which they showed an explicit awareness of which geometrical notions were embedded within each circle construction

In phase 2, the analysis of the children's choices between using intrinsic and Euclidean notions in constructing the task figures provided evidence of a balance in their use of both kinds of notions, i.e. the children seemed to be quite prepared to use both intrinsic and Euclidean ideas in planning and explaining turtle actions

In order to illustrate the children's use of both types of geometrical ideas, an example is given of an episode which took place during their solution of task 4, fig. 2. The episode illustrates how the children saw the connection between the positions of the centre points of the three circles. After having tried out several strategies involving the construction of one circle after the other and stumbling on the working out of the interface between the second and third circle, Alexandros seemed to spot the uniformity of the lengths between the circles' centres by revolving the piece of paper on which the figure was drawn, so that a bottom circle would go to the top and vice versa. Although the children seemed enthusiastic about their "discovery" concerning the distances between centres, they still did not consider the positions of the centres; although they had decided on the length of the second interface, they turned the turtle 45 degrees to the left apparently using their perceptual cues (fig. 3).

```
CIR19 40
RT 90
PU
FD 80
LT 90
PD
CIR19 40
LT 45
```

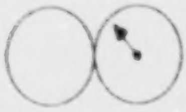


Figure 3 The children's perceptual strategy in task 4

The researcher decided to prompt them to focus their attention on the uniformity of the figure they had noticed from Alexandros' turning of the piece of paper. Their dialogue at this point illustrates how the relationship among the three centres was made explicit by the children,

> V "You know what I'm thinking? Why should it be 45? (the turn) You know why? Since, if we join the three dots a triangle is been done (formed) an equilateral"
> A "Equilateral"
> V "Eh?"
> A "And the sum of the angles of a triangle is 180?"
> V "Look. It goes forward. It goes left, you know how much? It goes left 360 divided by 3. So, how much is it? 3 120. It goes left 120 It goes forward and does the circle (she observes that the turtle's current heading is zero) 120 30 because I was thinking that it's like that, so 90 plus 30 (she types LT 30)" (brackets are used for the researcher's comments)

This new strategy involved a rather complicated but coherent use of both intrinsic and Euclidean notions. The reference to the two radii forming the sides of a triangle and the centres of the circles forming its vertices implies the use of Euclidean ideas. On the other hand, deciding on the turtle's turning after constructing each circle was based on a partitioning of a total turtle turn. Furthermore, Valentini's argument for turning the turtle left from a zero heading to face the top vertice of the triangle, was based on partitioning the turtle's turn into a 90 plus 30 degree turn.

As implied from the above example, the findings in phase 2 suggest that the children did not seem to find inherent qualitative differences between Euclidean and intrinsic ideas when they used them while employing their turtle schema. An occasion where the children actually expressed this view was during a semi - structured interview after they had solved task 5 (fig. 2), the children were comparing the CIR4 and CIR9 circle procedures (fig.1), having used the latter in constructing the small circles of the figure in task 5.

> V "...the CIR4 and the CIR9 are the same, because..."
> R "The same?"
> V "I mean that they are related in this shape in particular. I mean that it goes there, I turn left again, I give it a number. It does the circle again I turn it right and take it back"
> R "So what is it that makes them almost the same?"
> V "Right. That... of course in one we know the precise... (she means length of radius) in the other one we don't know it, but here in both cases we turn and we make the circle as usual, while if I said that CIR4 and CIR19 were the same they are not

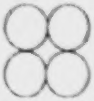
> the same because in CIR19 it starts from the middle like TC and in those two it starts from the edge"

In this occasion, Valentini seemed to refer to the CIR4 and CIR9 procedures as a product of the turtle's action, implicitly de - emphasising how this action is quantified. Her criterion for distinguishing the CIR4 and CIR9 procedures from the CIR19 and the TC involved the notion of where the turtle started (and ended) in constructing the circle, i.e. on the curve itself or on a point away from the curve. In explaining why she thought CIR4 and CIR9 were "equally easy", Valentini subsequently said:

> V: "Because both of them make a circle. A 36 - agon that is. Especially from the turtle's point of view, the turtle would say that 4 is easier. Because 4 is completely clear, you tell her 'go forward turn, go forward turn' while in CIR9 it does all that thing."
> R: "So, for the turtle CIR4 is easier. Does that mean that for you CIR9 is easier?"
> V: "It's the same."
> A: "It's the same."

The indications that the children's criteria for using Intrinsic and Euclidean notions were not primarily related to inherent characteristics of the notions themselves, but rather on aspects of the broader mathematical situations generated during the research, lead to a further prompting of which of these aspects were important in the forming of the children's choices and why.

Two factors concerning the role of the intrinsic schema in the children's choices emerged from the analysis. Firstly, they found employing the schema meaningful and did not seem to favour one kind of notion or the other as a consequence of having employed it. Secondly, their criteria for using the schema tended to relate to its intuitive nature rather than to a particular type of geometrical notions. The programming and modularity involved in the children's strategies also influenced their choices on which notions to use. However, their priorities in their decisions lay with the programming rather than with what kind of geometrical notions to use. The children's critical remarks on generalised rules involving Intrinsic and Euclidean notions referred to whether the rules had been derived via an inductive method or not, rather than on which kind of generalised rules were easier to understand. Finally, the children expressed a preference for employing notions which they had previously used in personally meaningful contexts rather than those presented to them through the school system Their distinction between "personalised" and "impersonal" notions, however, did not seem to be related to the distinction between intrinsic and Euclidean notions



REPEAT 4 [CIR19 20 PU FD 40 PD RT 90]

TASK 8

Figure 4 The children's solution of task 8

An episode encapsulating the above issues occured within the context of the children discussing their solution of a task - figure involving circles placed in a square formation (fig 4) Both children said that they preferred explaining the turtle's turn of 90 degrees by means of the partitioning of a total turn rather than adding up the internal angles of the formed square What is interesting is not their preference as such (after all, they were using the turtle to construct the figure), but the reasons they gave for and against the two methods

(The children's reasons on why they did not prefer the internal angles method)
V " they tell us, that definitely it's 360 (she means that the sum of the internal angles is 360 degrees) and that's it, you can't say anything, it's definitely 360, I know and you can't ask, you can't do a thing"
A "It's like I told you the other time Geometry forces us, we can't ask her this, since it's been discovered that this is that much, that much we'll write it We can't ask her why is it like that and why is it like this because they'll tell us because that's what it want's to be"

(Their reasons on why they preferred the total turn method)
V "Because it's more natural yes it's more natural, now I thought of that anybody can understand it "
A "Even if he doesn't know turtle at all"
R "Tell me something What does someone have to know to understand this thing?"
V "Nothing"

As implied by the above example, the children did not seem to consider one type of notions easier to understand than the other Instead, their choice of which one to use was influenced by other factors of the generated mathematical situations outlined above, such as whether using a notion had been part of the children's personal experience Therefore, it was ultimately those factors which were important in the children's choices and not whether one type of notions, Intrinsic or Euclidean, made more sense than the other

CONCLUSIONS

The analysis has indicated that the children did not find qualitative differences between the nature of Intrinsic and Euclidean notions while employing their turtle schema in the Circle microworld The case - study can therefore be used to argue that there is rich educational potential in creating microworld environments which on the one hand invite children to use their intrinsic schema and on the other embody conceptual fields (in the sense of Vergnaud, 1982) incorporating a range of geometrical ideas substantially wider than the one provided by Intrinsic geometry

ABELSON, H. & Di SESSA, A., (1981), Turtle Geometry The Computer as a Medium for Exploring Mathematics, MIT Press

HILLEL, J., & KIERAN, C., (1987), Schemas Used by 12 year-olds in Solving Selected Turtle Geometry Tasks, Recherches en Didactique des Mathematiques

HILLEL, J., ERLWANGER, S., & KIERAN, C., (1986), Schemas used by Sixth Graders to Solve Turtle-Geometry Tasks, Research Report No 3, Concordia University, Mathematics Department

HOYLES, C., & SUTHERLAND, R. (in press) Logo Mathematics in the Classroom, Routlege

HARVEY, B., (1985), Computer Science Logo Style, Vols 1 & 2, MIT Press

KYNIGOS, C. (1989), From Intrinsic to Non - Intrinsic Geometry A Study of Children's Understandings in Logo - Based Microworlds, Unpublished Doctoral thesis, Institute of Education University of London

KYNIGOS, C. (in press) The Turtle Metaphor as a Tool for Children Doing Geometry, M.I.T. press

LAWLER, R.(1985)Computer Experience and CognitiveDevelopment,Ellis Horwood

LERON, U., (1983), Some Problems in Children's Logo Learning, Proceedings of the Seventh International Conference for the Psychology of Math Educ 346-352

NOSS, R., (1985), Creating a Mathematical Environment through Programming A Study of Young Children Learning Logo, Doctoral Thesis, published by University of London Institute of Education

VERGNAUD, G., (1982), Cognitive and Developmental Psychology and Research in Mathematics Education Some Theoretical and Methodololgical Issues, For the Learning of Mathematics, 3, 2, p31-41

MATHOPHOBIA: A CLASSROOM INTERVENTION AT THE COLLEGE LEVEL

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Following an investigation conducted with mathophobics students (Gattuso, Lacasse,1986), we formulated a set of working hypotheses for mathematics teaching. This new model was put to use in a regular class of at the college level. The main objective was to create an environment in which the affective aspects of learning mathematics would be recognized and coped with, along the lines determined by our former research, through genuine mathematical activities. (This workshop will be presented in English)

1 Cheminement

Ce texte présente les résultats d'une recherche menée en vue d'articuler un modèle d'intervention en classe qui permette à l'élève de poursuivre des activités mathématiques dans un contexte qui minimiserait l'émergence de réactions négatives face à cette démarche. Nimier (1976), Tobias (1978), Blanchard-Laville (1981) et Blouin (1985,1986,1987), entre autres, font apparaître l'importance du côté affectif de l'enseignement des mathématiques; nous nous intéressons particulièrement à ces aspects négatifs qui sont recouverts par le terme "mathophobie". Notre modèle devait s'inspirer des résultats obtenus dans notre recherche : "Les mathophobes: une expérience de réinsertion au niveau collégial." (Gattuso, Lacasse 1986).

Cette première étude nous a permis d'analyser le phénomène de la mathophobie dans le cadre d'ateliers qui avaient pour but de réconcilier un certain nombre d'élèves ayant un vécu négatif face aux mathématiques. (Lacasse,Gattuso, PME XI, 1987). À l'origine, il s'agissait de voir s'il était possible de modifier l'attitude des élèves face aux mathématiques.

Les résultats de notre première recherche auprès des mathophobes sont regroupés autour de treize énoncés ou hypothèses qui semblent nous indiquer un ensemble de conditions permettant de créer un environnement favorable à l'apprentissage des mathématiques, du moins en ce qui concerne l'aspect affectif. Par contre, ces hypothèses avaient été générées dans un cadre bien spécifique: celui des ateliers "Phobie des maths". C'est ainsi que nous avons été amenés à prévoir un deuxième volet à cette recherche. L'analyse des résultats obtenus a mis en évidence un certain nombre de facteurs sur lesquels les enseignants peuvent effectivement intervenir dans une démarche didactique régulière. Une partie des résultats a été présentée au PME XII (1988). Nous aimerions maintenant compléter cette présentation, à la lumière de l'analyse que nous avons faite depuis.

2 Les problèmes d'implantation

Le commencement

La question cruciale, au début, c'était bien sûr de construire le cours en tenant compte des hypothèses. Nos avions fait le choix d'une démarche très orientée vers l'activité de l'élève. Or, l'environnement physique au sens large nous amène beaucoup de contraintes. Lors des activités, les élèves travaillent en groupe mais les locaux sont exigus et les tables de travail sont assez petites. De plus, la disposition n'est pas prévue pour ce genre de travail. Lors de notre intervention, le hasard nous avait attribué des classes qui étaient libres à la suite des périodes de cours. Les élèves en profitent souvent pour rester en classe après le cours et poursuivre leur travail ou leurs échanges. L'atmosphère demeure très détendue. Dès le départ les relations personnelles entre les élèves et l'enseignante sont favorisées. La présence au cours s'en est ressentie de même que la complicité entre les élèves.

La construction des activités

Nous avons pu expérimenter plusieurs activités utilisant du matériel concret. La plupart du temps, le matériel est bien reçu des élèves, même s'il n'est presque jamais destiné au collégial. L'enseignant doit constamment inventer, imaginer et aussi construire des supports concrets que les élèves peuvent manipuler. Il y a beaucoup de travail à faire de ce côté. Nous avons pu constater lors de la préparation des protocoles d'activités, que trop souvent dans l'enseignement certains concepts implicites sont pris pour acquis. Il est nécessaire qu'ils soient explicités. Cela est surtout vrai dans le contexte de résolution de problèmes. L'enseignant doit insister sur la démarche de résolution de problèmes et non seulement sur la solution ou sur le contenu. Lors d'un cours à caractère non magistral, ce sera, entre autres choses, par ses questions que l'enseignant pourra mettre en valeur ses processus, ses démarches et surtout celles de ses élèves.

Les activités libres ont donné aux élèves l'occasion d'émettre des hypothèses, d'échanger des résultats et de se poser d'autres questions. De plus, ils apprécient le fait de découvrir, cela leur permet de comprendre. Cependant, il y a tout un apprentissage à faire pour eux et ce n'est pas évident. Ils ont des difficultés à travailler, à se développer une méthode. C'est pourquoi le rôle de l'enseignant est très important. Il doit agir comme un guide: resituer l'élève, relancer le travail, poser des questions et éclairer à l'occasion mais sans rien imposer.

3 L'évolution des élèves: attitudes et comportements

La classe régulière.

A la suite de l'expérience des ateliers, nous nous posions une question fondamentale: est-ce possible de mettre en place cette façon de travailler avec les élèves réguliers? Or, bien qu'il soit difficile de voir des résultats immédiats sur le strict plan de la performance, dans l'ensemble, nous répondons oui à cette question pour plus d'une raison.

Il y a eu sensiblement moins d'abandons, soit sur le plan formel (abandon de la session), soit sur le plan de l'activité quotidienne. Lors des entrevues, nous avons pu constater l'effet généralement positif de notre approche. Les élèves expriment avec grande facilité leurs réactions parfois négatives, parfois positives par rapport au cours, aux mathématiques.

Le phénomène du déblocage chez les élèves.

Nous avons perçu, en cours de session, un changement chez certains élèves, un déblocage par rapport aux mathématiques. Pou l'élève, il s'agit de se dire: je vais réussir ce cours de mathématique et je vais prendre les moyens nécessaires à cette réussite. Le déblocage peut être identifié à un moment, très précis; c'est-à-dire, il peut y avoir une période de gestation et puis, tout d'un coup, on trouve qu'on a du plaisir à faire des mathématiques. Cependant, il ne faut pas penser que c'est permanent. C'est plutôt un zig zag; mais, une personne qui vit une expérience positive en tire une certaine capacité à faire face à la prochaine attaque d'anxiété de façon un peu plus solide.

Le vécu des élèves: les activités ouvertes et la communication.

Dans les communications qu'on veut établir dans une classe, on cherche à faire en sorte que les élèves partagent leur vécu mathématique. Evidemment, il faut leur faire vivre quelque chose, parce que bien souvent leur passé mathématique est réduit au minimum. C'est le rôle des activités ouvertes. Ce style d'enseignement qui consiste à ne pas trop donner de réponses mais à relancer le questionnement s'avère plus efficace probablement lorsqu'il est utilisé, comme on l'a fait, de façon très systématique.

Deux facteurs sont gênants dans la poursuite de cette démarche. Premièrement, il y a le programme, le système. C'est une énorme contrainte; les objections des enseignants à propos du programme à suivre sont compréhensibles. Il y a aussi le facteur relié au temps qu'il faut pour démarrer. Au début, l'incertitude est totale. On se demande si les élèves qui vont avoir réalisé telle partie des activités vont avoir réussi à couvrir tous les éléments.

Selon les dossiers scolaires, dix élèves sur 19 ont été récupérés temporairement. On peut donc présumer que ce modèle d'enseignement favorise l'apprentissage des mathématiques. Le taux de réussite assez faible de la session suivante permet de supposer que l'intervention est positive mais peut-être pas assez longue. Un suivi serait nécessaire. Il fallait s'attendre à cette conclusion: il est impossible de refaire en une session un mode d'apprentissage qui s'est installé pendant des années. Dans l'ensemble, nous croyons qu'une étude plus complète du cheminement des élèves devrait être faite et devrait mettre en jeu un suivi sur plusieurs sessions. A la lumière de notre travail dans les ateliers de phobie des maths et à la suite de nos deux expérimentations, nous croyons que la prochaine étape, c'est de suivre un groupe d'élèves à travers tous leurs cours de mathématiques au cégep. Nous pourrions alors mesurer la qualité de l'intervention et la permanence des acquis sur toute la durée de leur présence au cégep.

4 Le jeu des relations et des perceptions

Canaux de communication

La forme de travail privilégiée dans notre intervention a été le travail d'équipe. L'enseignante a mis les élèves dans une situation favorisant les échanges et le travail de groupe. Nous y avons vu plusieurs avantages. Le travail en groupe permet une activité mathématique qui se trouve souvent enrichie par des discussions qui émanent des questions des élèves. Certains d'entre eux ont pu prendre confiance en eux et se sont senti valorisés de pouvoir en aider d'autres. De plus les situations de résolution de problèmes permettent à certains élèves, et ce ne sont pas les forts habituels, de se faire valoir; en effet, ces situations font souvent appel à d'autres habiletés que celles généralement utilisées dans nos classes de mathématiques comme l'imagination, la capacité de synthèse, la vision globale.

Le travail en groupe, nous avons pu l'observer, comporte cependant ses difficultés. Les très faibles et trop timides ne s'intègrent pas à une équipe, d'autres ont tendance à trop se fier sur les voisins. Par ailleurs certains élèves plus forts n'ont pas toujours envie d'aider les plus faibles. Ils mettent beaucoup de temps à travailler leurs mathématiques et voudraient avec raison que les autres en fassent autant. Ils ne veulent pas perdre leur temps.

Pour les élèves au point de départ, verbaliser leur démarche n'est pas facile. Mais le travail en équipe, surtout au moment d'activités plus exploratoires, amème doucement les élèves à décrire ce qu'ils font et ensuite ils en discutent et l'évaluent. Ce procédé les aide à comprendre ce qu'ils font (que ce soit juste ou non) trop souvent instinctivement sans aucune analyse. Ils sortent de ces échanges valorisés. L'enseignant qui les écoute peut juger des acquis, même si, à l'occasion, certains élèves restent insécures.

Relations élèves-enseignant

Si l'apprenant prend conscience que les résultats découlent de son travail, il y a de bonnes chances qu'il puisse les retrouver au besoin. L'eurêka, c'est simplement un signal très important que la personne reste accrochée à sa démarche. Cette réaction est aussi une forme d'auto-renforcement ou de renforcement interne et c'est dans ce sens que le fait d'apprendre nous rend curieux et nous pousse à vouloir apprendre autre chose. L'enseignant cherche donc à multiplier les occasions d'émergence de ce signal qui ne peut survenir que dans les activités.

Les élèves prennent plaisir à comprendre et à réussir. Ils sont par la suite plus encouragés à poursuivre leur travail. D'autre part, l'enseignant doit faire en sorte que les acquis soient conservés. Pour cette raison et également pour favoriser une certaine continuité dans les cours, il nous apparaît primordial qu'il souligne les découvertes des élèves soit individuellement ou en groupe à l'occasion d'un retour à la fin d'une période de travail.

Un dernier point: le cours magistral sert aussi dans une bonne mesure à transmettre le vécu de l'enseignant. Il est important que l'enseignant puisse montrer ses démarches, ses conjectures, ses tâtonnements. Dans notre cas, comme il y a eu très peu de cours magistraux, les occasions de tranmettre ce vécu se sont faites rares. Cependant, nous restons avec le sentiment que cette dimension doit demeurer présente à l'esprit de l'enseignant comme si le choix de cette formule didactique était conditionné par le désir de transmettre une partie de son vécu mathématique.

5 Le vécu de l'enseignant

Premières constatations

Nous avons prêté une attention particulière au cheminement de l'enseignante et il nous a semblé que les points suivants étaient les plus saillants.

D'abord la personnalité de l'enseignante, son "personnage" en quelque sorte, est reliée à la gestion du cours comme tel. Par exemple, le fait de se rendre disponible entre les cours a pu jouer dans le déblocage de certains élèves. Mais cette disponibilité très grande et la réponse que les élèves y font dépend dans une large mesure de personnalités qui s'accomodent. Le rep test fait état d'une composante importante pour l'enseignante: la capacité pour l'élève d'utiliser la ressource "prof". Un manque de maturité de l'élève sera mal reçu ou mal perçu sur ce plan.

Deuxièmement, la structuration du cours, surtout dans les parties les plus novatrices, pose des problèmes. Il faut souvent créer du matériel nouveau selon une démarche inhabituelle puisqu'expérimentale. La pression constante des régularités du milieu engendre une fatigue qui, assez rapidement, va faire en sorte que l'enseignant va retomber dans ses anciens schémas, ses anciennes habitudes. Il aura recours à des méthodes d'enseignement plus traditionnelles, des méthodes où les élèves, comme l'enseignant, ne sont pas constamment confrontés à la recherche du sens.

Les conceptions des élèves eux-mêmes sont parfois des obstacles sérieux à ce nouveau fonctionnement. Ce sont toutes les fausses conceptions qui jouent: les élèves habitués à d'autres exigeances s'accomodent mal de ce nouveau rôle surtout s'ils croient qu'il ne se poursuivra pas dans les cours suivants.

Enfin, le modèle applicable semble s'orienter vers deux aspects principaux: un enseignement visant surtout la communication où l'importance est dans la construction et l'utilisation du langage mathématique et un enseignement axé sur la découverte où le contenu mathématique est primordial. Ces deux aspects doivent être conservés à travers une étape de consolidation formelle des acquis. Cette consolidation dépend de l'action de l'enseignant, par exemple au niveau de la clôture de chaque cours ou de chaque séquence d'activités.

Les exigences en termes de disponibilité de l'enseignant

L'un des avantages de la méthode de travail par activités devrait être de permettre à l'enseignant d'intervenir plus précisément selon les besoins de chacun.

Nous avons pu constater qu'il est possible d'agir ainsi en classe même en dépit des contraintes.

Par ailleurs, les élèves ont souvent eu recours à l'aide de l'enseignante après les périodes de classes. Les élèves sentent qu'il y a toujours une possibilité de demander de l'aide soit à l'enseignant soit aux autres élèves. Ils arrivent ainsi à dépasser leur timidité à poser des questions car, en général, ils n'y sont pas habitués et ils ont toujours peur de paraître ridicule ou stupide. La préparation par l'enseignant de ce qu'on pourrait appeler l'environnement didactique doit donc inclure cette composante.

En guise de conclusion

Dans l'état actuel des choses, les faiblesses des élèves se situent sur deux plans: il y a l'aspect préparation au contenu mathématique et il y a la préparation à travailler tout court. En tant qu'enseignant en mathématiques, nous trouvons qu'il y a quelque chose de spécial à l'intérieur de l'activité mathématique. Ce n'est pas nécessairement relié aux notions apprises à l'école; dans la vie de tous les jours, quand se sert-on des mathématiques offertes au niveau collégial? Mais, dans la vie courante, la formation mathématique conditionne-t-elle notre façon de réagir à différentes situations? Peut-être que oui, mais comment? N'est-ce pas dans cette direction qu'il faut chercher une raison d'être à l'enseignement des mathématiques à quelque niveau que ce soit?

Références

BLANCHARD-LAVILLE, C. (1981). "Les dimensions affectives de l'apprentissage des statistiques", Education permanente, no 61, 41-62.

BLOUIN, Y. (1985). La réussite en mathématiques au collégial: le talent n'explique pas tout. Cégep François-Xavier-Garneau.

BLOUIN, Y. (1986). Réadapter les handicapés des mathématiques. Prospectives, 22 octobre, 115-121.

BLOUIN, Yves (1987). Éduquer à la réussite en mathématiques. Cégep François-Xavier-Garneau.

GATTUSO, L. et LACASSE, R. (1987) Les mathophobes une expérience de réinsertion au niveau collégial. Actes du onzième congrès international de Psychology of Mathematics Education, PME-XI, édité par Jacques C. Bergeron, Nicolas Herscovics, Carolyn Kieran. Juillet: Montréal.

GATTUSO, L. et LACASSE, R. (1988) Intervention in a mathematics course at the college level. Actes du douzième congrès international de Psychology of Mathematics Education, PME-XII, . Vesprem.

GATTUSO, L. et LACASSE, R. (1986). Les mathophobes une expérience de réinsertion au niveau collégial. Cégep du Vieux Montréal.

NIMIER, J. (1976). Mathématiques et affectivité. Stock.

TOBIAS, S. (1980). Le mythe des maths, traduit par Romain Jacoud. Paris-Montréal: Etudes vivantes.

LE MICRO-ORDINATEUR, OUTIL DE REVELATION ET D'ANALYSE DE PROCEDURES DANS DE COURTES DEMONSTRATIONS DE GEOMETRIE.

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Résumé. Les élèves français de 12 à 14 ans mettent en oeuvre leur raisonnement déductif principalement en géométrie. Les difficultés rencontrées sont très importantes ; elles hypothèquent quelquefois la suite de leur scolarité mathématique. L'étude présentée ici vise à connaître l'origine et la nature des erreurs les plus fréquentes. Nous utilisons pour cela le micro-ordinateur qui s'avère puissant outil de révélation et d'analyse des démarches des élèves, en particulier dans le cas où l'activité déductive se réduit à une simple inférence. Des méthodes statistiques multidimensionnelles permettent de dégager les grandes structures de comportements erronés.

Abstract. French pupils, between the ages of 12 and 14 use deductive reasoning especially in geometry. They have to cope with many difficulties which may jeopardize their success in future mathematics courses.The study that we are submitting here aims at a deeper knowledge of the origin and the nature of the most common mistakes. In order to achieve this we use the micro-computer which appears to be a powerful tool to reveal and analyse pupils'ways of reasoning, especially when the deductive activity is limited to a simple inference. Multi-dimensional statistical methods provide us with the possibility of bringing out the main structures of erroneous behaviours.

§ 1 - PROBLEMATIQUE.

Des observations et quelques études plus approfondies de productions d'élèves, de $4^{ème}$ en particulier (13-14 ans), sur les problèmes à démonstration géométrique, ont montré la multitude et la grande variété des procédures erronées des élèves. Certes, les erreurs puisent leur origine profonde dans l'absence de signification de la preuve mathématique et dans une carence de maitrise du lexique nécessaire (*puisque*, *donc*, *or*, *car* ...), mais également de façon ou conséquente ou conjointe :

* dans une absence de rigueur dans l'articulation dissymétrique des trois éléments-clés de l'inférence : hypothèse - théorème - conclusion

* dans la prise en compte d'indicateurs extrinsèques pour choisir l'un quelconque de ces éléments-clés :

. indicateurs formels (structure, rythme, ...)
. " sémiotiques (mot, lettre, symbole, ..)
. " sémantiques (un sens voisin, une utilisation antérieure, ...).

Il est difficile, voire impossible, pour l'enseignant de repérer à chaque fois dans une copie d'élève le type d'erreur commise et surtout sa répétition chez l'élève, sa fréquence dans la classe et les conditions dans lesquelles l'erreur s'élabore et apparait. De plus, il lui est encore plus difficile de trouver pour chaque élève les situations qui permettraient de perturber et mieux, d'éliminer les procédures erronées.

L'ordinateur, en échange, permet un travail plus individualisé et, surtout, une sanction immédiate de l'erreur et donc un retour de l'élève sur ses procédures.

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(1) Professeur au Lycée Ile-de-France - Rennes.

(2) Professeur à l'I.R.E.S.T.E. - Université de Nantes.

§ 2 - METHODOLOGIE RETENUE.

Il semble donc important, pour mieux traiter ensuite ces procédures chez chaque élève, de les identifier et d'en repérer les circonstances d'apparition. Il parait nécessaire de limiter les variables en interaction et pour cela de fournir à l'élève des situations où le sens entretenu par le but lointain de la démonstration n'est pas le moteur essentiel et où le lexique est réduit.

Pour ce faire, on établira une liste de faits mathématiques (géométriques en l'occurence) pouvant tenir lieu, suivant les situations, d'hypothèses ou de conclusions et une liste de théorèmes. Une inférence incomplète - voire un problème à démonstration - étant proposée, l'élève devra choisir un ou plusieurs faits, un ou plusieurs théorèmes pour que soit validées l'inférence ou les inférences successives. La tâche de l'élève sera exécutée à l'aide d'un logiciel permettant un travail personnel, puis une analyse individuelle de ses réponses (après éventuellement 2 essais).

Notre tâche didactique et informatique [1] consistera alors, à plus ou moins long terme :

* à construire des situations où les variables sont contrôlables ;

* à identifier et interpréter les erreurs et les conditions de leur émergence ;

* à construire un modèle prédictif de procédures erronées ;

* à construire des situations où celles-ci seraient déséquilibrées ;

* à élaborer des logiciels satisfaisant les objectifs didactiques.

Schématiquement, compte tenu de ces objectifs, le micro-ordinateur est intégré sous 2 aspects :

* aide tutorielle de l'élève dans une situation de problème à démonstration (logiciel D)

* aide pour l'enseignant à mieux comprendre les erreurs commises par l'élève et donc si possible à les corriger (logiciels "Premiers Pas" et "Multipas") .

L'évaluation du logiciel D (aide à la démonstration), souligne, entre autres, trois difficultés :

1°) Les élèves, dans la conduite de la démonstration, butent sur des obstacles de nature *logique* : difficultés à identifier avec précision
- la ou les hypothèses associées à une assertion restant à prouver
- la ou les conclusions découlant d'hypothèses données et d'un théorème
- le théorème justifiant que telle hypothèse conduit à telle conclusion.

2°) Les obstacles rencontrés par l'élève sont aussi très souvent d'ordre *lexical* et *discursif*.

3°) Le nombre de variables didactiques à contrôler est élevé et certaines d'entre elles demeurent difficilement maîtrisables. Aussi, pour échapper à un empirisme préjudiciable à la recherche-même, nous allons chercher, à travers une recherche incidente, à limiter les variables en jeu afin d'agir plus efficacement sur elles.

Nous sommes alors conduits de façon nécessaire à affronter différentes questions liées à ces trois difficultés. Comment venir à bout de celles-ci ? Comment aider les jeunes élèves ($5^{\text{ème}}$ et début $4^{\text{ème}}$) dans l'apprentissage de la démonstration, en commençant par celle à un pas, pour éliminer les difficultés introduites par la rédaction et la conception globale d'un problème ?

---

(1) Dans le cadre du Groupement de Recherches du C.N.R.S. : "Didactique et Acquisition des connaissances scientifiques". Le groupe de Rennes est constitué, outre les présentateurs de ce texte Marie-Danièle Fontaine (Collège de Combourg), Alain Nicolas (L.E.P. Victor-Rault), André Simon (Lycée Bréquigny), d'I. Giorgiutti, F. Ruamps (Institut Mathématique de Rennes), de P. Nicolas et D. PY (Institut de Recherche en Informatique et Systèmes Aléatoires de Rennes) et de C. Boulard (Collège La Harpe, Rennes). Tous ces enseignants-chercheurs participent à cette recherche, à son expérimentation et son évaluation.

Comment leur permettre de savoir faire un choix pertinent, parmi une liste d'assertions et de théorèmes, de triplets dont les termes sont :

- hypothèse
- théorème
- conclusion ?

Exemple : Questionnaire : 6 questions indépendantes. Hypothèses et théorèmes sont donnés. La conclusion est à trouver. (cf. analyse § 4).

Nous avons entrepris pour ce questionnaire le traitement statistique des données recueillies suivant deux méthodes d'analyse : la classification hiérarchique (selon I.C. LERMAN) et la classification implicative (selon R. GRAS). Nous verrons plus loin les résultats que nous en avons déduits.

D'ores et déjà, nous pouvons nous demander sur quoi s'appuie la stratégie de décision de l'élève dans cet exercice très particulier qui consiste à faire un choix parmi un ensemble fermé de solutions ?

Cette stratégie est nécessairement fort proche de celle déployée dans les Q.C.M., et, en revanche, très différente de celle qui est suivie dans les démonstrations à plusieurs pas, dans les problèmes ouverts et même dans le logiciel D. Ici l'élève doit seulement retenir ou rejeter un élément d'une liste. Il n'a pas de véritable activité créatrice. De plus, le sens global n'est pas mobilisable ; les seuls points d'appui sont le sens du pas de démonstration et l'ensemble langagier des assertions ou théorèmes dont il dispose. Nous avons cependant remarqué, grâce à la répétition, à l'accumulation et à la concomitance d'erreurs, la stabilité de certaines procédures qui correspondent à des modèles de fonctionnement en équilibre aussi bien chez un élève particulier que chez l'élève en général. Les erreurs, que nous appelons tous "erreurs de raisonnement", relèvent de causes profondément ancrées et pas seulement d'ordre logique. Elles tiennent aussi à la méconnaissance des objets traités (quand ce n'est pas du vocabulaire utilisé) et aussi, très fortement, lors de l'articulation *hypothèse* $\xrightarrow{\textit{théorème}}$ *conclusion* , au pouvoir attracteur de certains mots, certains signes ou symboles, certaines formes (structures de phrases, rythmes,...). L'élève assemble plus, quand il se trompe à partir d'un critère "signe" que d'un critère "sens". Il va puiser dans les solutions offertes les indices formels les plus vraisemblables, les plus pertinents pour lui.

§ 3 - LOGICIELS DE REVELATION ET D'ANALYSE.

3.1. "Premiers Pas".

Il ne s'agit pas à proprement parler d'un didacticiel mais plutôt d'un outil de diagnostic.

a) Le module élève :

L'élève dispose d'une liste de faits (énoncés pouvant servir aussi bien en hypothèse qu'en conclusion) et d'une liste de théorèmes repérés par un numéro. Une démonstration à un pas - inférence - lui est proposée. Elle comporte un ou plusieurs trous qu'il doit compléter. Toutes les réponses fournies sont bien entendu conservées.

Exemples de questions. 1 et ?.....3.....? il manque une hypothèse et la conclusion
?.....2..... 5 seule l'hypothèse est demandée (elle peut comporter un ET).

Suivant le choix fait au départ par l'enseignant (module PREPA) l'élève dispose de plusieurs essais ou non et la bonne réponse lui est donnée ou non.

b) Le module PREPA : préparation du professeur.

Outre quelques options que le professeur peut choisir (cf. ci-dessus), l'essentiel du travail de préparation est la constitution des fichiers de faits, théorèmes, démonstrations, exercices.

Le fichier "démonstrations" contient les inférences "exactes" attendues par l'enseignant et le fichier "exercice" localise les "trous".

Il est à noter que l'enseignant a l'entière liberté de son exercice, tant du point de vue du choix des théorèmes et faits que de leur formulation. Il a l'entière responsabilité du choix des questions en fonction des variables didactiques qu'il souhaite observer. Le logiciel est donc parfaitement neutre de ce point de vue et personnalisable en fonction d'objectifs :

1. Renforcement d'apprentissage du fonctionnement dun pas déductif.
2. Bilan, recensement des acquis des élèves.
3. Révélation, analyse, diagnostic des erreurs pour une étude didactique.

c) Le module BILAN.

Il comporte des compteurs standards gérant les fautes les plus courantes, comme inversion hypothèse-conclusion, et des compteurs non standards qui permettent au professeur d'étudier de façon plus précise des variables didactiques.

3.2. MultiPAS.

a) Objectifs.

Comme nous l'avons vu, la vocation essentielle de "Premiers PAS" est le diagnostic des procédures d'erreur commises par les élèves. Cependant son utilisation nous fait découvrir d'indéniables apports au niveau de l'apprentissage de la démonstration à un pas.

L'ambition de "MultiPAS" est de mettre plus l'accent sur l'objectif apprentissage : il sera proposé aux élèves de résoudre un problème simple mais complet, avec à sa disposition :

- * des faits-données
- * un fait-conclusion
- * une liste de théorèmes
- * des faits "intermédiaires".

L'opérationnalisation de cet objectif se poursuivra suivant deux axes :

1 - la reconnaissance du changement possible dans le statut d'un fait (un fait démontré, qui apparait en conclusion d'un pas, peut être utilisé comme hypothèse ou partie d'hypothèse dans un pas qui suit) ;

2 - l'enchainement des pas, avec la possibilité donnée éventuellement à l'élève d'inscrire ses pas dans l'ordre de son choix, y compris à partir de la conclusion.

b) Conception générale.

"MultiPAS" hérite du logiciel précédent une conception en trois modules : * préparation des exercices par l'enseignant ;

* recherche d'une démonstration par les élèves (toutes les réponses sont enregistrées) ;

* bilan, exécuté après le passage des groupes d'élèves.

§ 4 - ANALYSES STATISTIQUES ET DIDACTIQUES D'UN QUESTIONNAIRE.

4.1. Présentation du questionnaire.

Un ensemble de 6 questions est proposé à des élèves de la classe de 5ème (12-13 ans) après l'enseignement de quelques propriétés de la symétrie par rapport à un point. A chaque question correspond une inférence que l'élève doit compléter en choisissant un des 11 faits suivants à titre de conclusion :

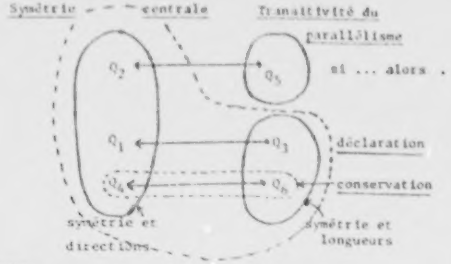
```
***************
FAITS
***************

1       (EF) ET (CD) SONT SYMETRIQUES PAR RAPPORT AU POINT I
2       [MN] EST LE SYMETRIQUE DE [PR] PAR RAPPORT AU POINT I
3       (AB) ET (CD) SONT SYMETRIQUES PAR RAPPORT AU POINT O
4       (MN)//(PR)
5       (CD)//(EF)
6       (AB)//(CD)
7       (AB)//(EF)
8       MN=PR
9       CD=EF
10       AB=CD
11       AB=EF


***************
THEOREMES
***************


1       LA SYMETRIE CENTRALE CONSERVE LES LONGUEURS
2       SI (D)//(D') ET (D')//(D'') ALORS (D)//(D'')
3       LE SYMETRIQUE D'UNE DROITE (D) PAR RAPPORT A UN POINT EST UNE DROITE (D')
 PARALLELE A (D)
4       SI DEUX DROITES SONT SYMETRIQUES PAR RAPPORT A UN POINT ALORS ELLES SONT
PARALLELES
5       DEUX SEGMENTS SYMETRIQUES PAR RAPPORT A UN POINT ONT MEME LONGUEUR
6       LA SYMETRIE CENTRALE CONSERVE LES DIRECTIONS
```

__Question__. Hypothèse et théorème des listes ci-dessus étant donnés, trouver la conclusion tirée de la liste des faits (2 essais sont possibles à chaque question).

DEMONSTRATIONS

	HYPOTHESES	THEOREME	? CONCLUSION à trouver
Q_1 Hypothèse : 1 Théorème : 3 → Conclusion : 5	(EF) et (CD) symétriques par rapport à I	Le symétrique de (D) par rapport à un point est (D')//(D)	(EF)//(CD)
Q_2 Hypothèse : 3 Théorème : 4 → Conclusion : 6	(AB) et (CD) symétriques par rapport à O	Si 2 droites sont symétriques par rapport à un point alors elles sont //	(AB)//(CD)
Q_3 Hypothèse : 2 Théorème : 5 → Conclusion : 8	[MN] est symétrique de [PR] par rapport à I	2 segments symétriques par rapport à un point ont même longueur	MN = PR
Q_4 Hypothèse : 3 Théorème : 6 → Conclusion : 6	(AB) et (CD) symétriques par rapport à O	La symétrie centrale conserve les directions	(AB)//(CD)
Q_5 Hypothèse : 6ET5 Théorème : 2 → Conclusion : 7	(AB)//(CD) et (CD)//(EF)	Si (D)//(D') et (D')//(D") alors (D)//(D")	(AB)//(EF)
Q_6 Hypothèse : 2 Théorème : 1 → Conclusion : 8	[MN] est symétrique de [PR] par rapport à I	La symétrie centrale conserve les longueurs	MN = PR

On peut schématiser les proximités formelle, sémantique et référentielle, a priori, de ces six questions :


Symétrie - - - centrale
Transitivité du parallélisme
si ... alors .
Q_2
Q_5
Q_1
Q_3
Q_4
Q_6
déclaration
conservation
symétrie et directions
symétrie et longueurs

4.2. Résultats.

1°) Paramètres des réussites.

a) Moyennes.

On retrouve la hiérarchie présumée a priori entre les réussites R_i aux 6 questions : R_2 (96,25 %), R_1 (78,75 %) et R_4 (72,5 %).

On a : $R_3 = R_6$ (87,5 %)

R_4 nettement inférieur à R_6, R_2 légèrement supérieur à R_3.

Le taux de réussite de Q_5 (85 %) est un peu inférieur aux taux de réussite de Q_3 et Q_6 (Q_5 ne fait pas référence à la symétrie centrale ; son théorème est instancié). Il est nettement inférieur à celui de Q_2 malgré la même formulation du théorème en "*si... alors...*" ; est-ce en raison de la double hypothèse ?

b) Coefficients de corrélation et χ^2 entre les modalités "réussites" des 6 questions.

- Les plus fortes liaisons positives sont observées entre :

R_1 et R_2 (formulation différente du théorème mais même contenu) : $\rho = 0{,}38$.

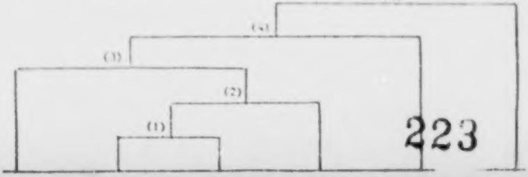
R_1 et R_6 ($\rho = 0{,}358$) : est-ce que ce sont les mêmes élèves qui ont des difficultés à la mise en train (Q_1) et à soutenir leur attention (Q_6) ?!

- R_3 et R_5 ont avec toutes les autres réussites un coefficient de corrélation très proche de 0 et même négatif sauf avec R_4.

2°) Analyse hiérarchique des réponses.

Nous utilisons la méthode de classification de I.C. Lerman selon l'algorithme dit de la vraisemblance du lien.

Arbre hiérarchique des réussites.



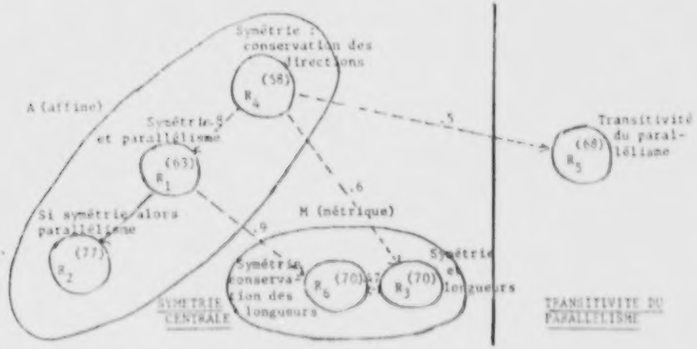
L'arbre complet figure en annexe.

7.4. Analyse implicative.

Selon une méthode analogue à celle de I.C. Lerman, R. Gras mesure l'implication entre attribut *a* et *b* à partir de l'indicateur de base $E_{a \wedge \bar{b}}$, ensemble des individus contredisant $a \Rightarrow b$.

Le tableau des implications permet de construire le graphe orienté, transitif, pondéré, associé à la relation de quasi-implication.

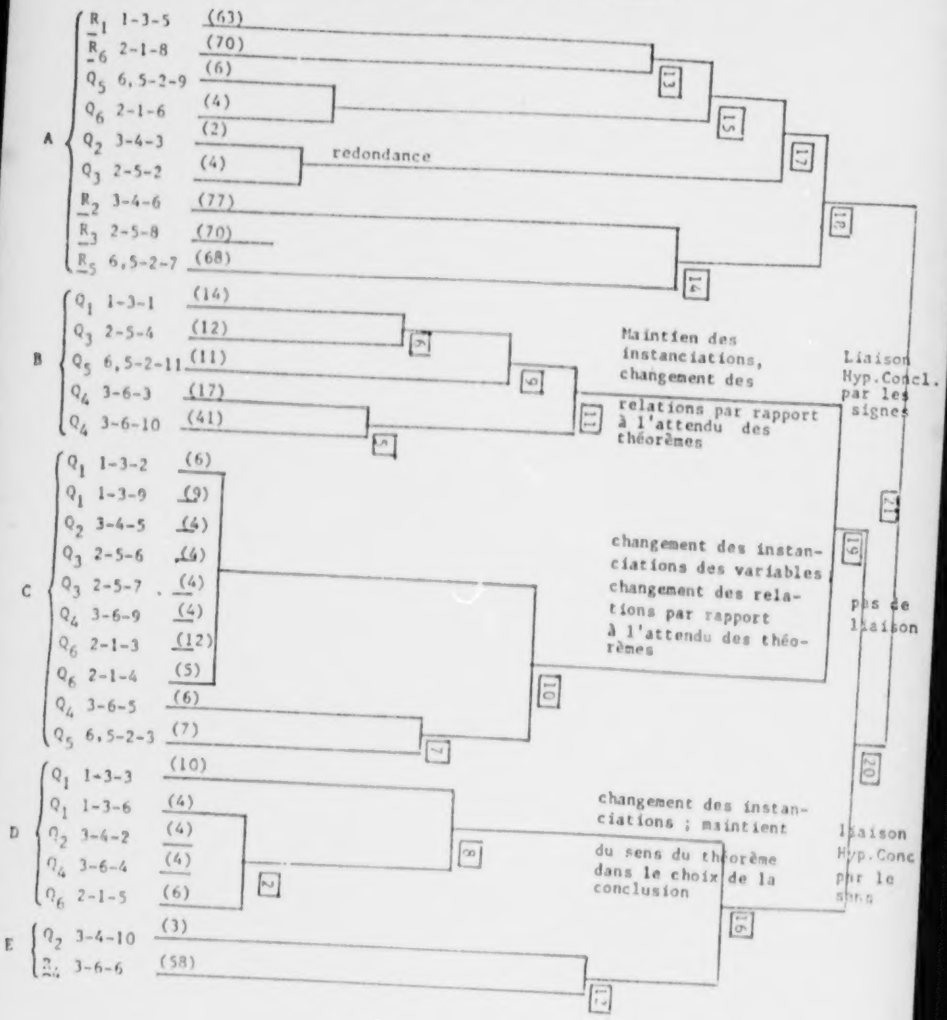
Arbre implicatif de réussites.


Symétrie :
conservation des
directions
(58)
R4
A (affine)
Symétrie .8
et parallélisme
(63)
R1
Si symétrie alors
parallélisme
(77)
R2
.9
.6
.5
M (métrique)
Symétrie
conserva-
tion des
longueurs
(70)
R6
.7
(70)
R3
Symétrie
et
longueurs
SYMETRIE
CENTRALE
Transitivité
du paral-
lélisme
(68)
R5
TRANSITIVITE DU
PARALLELISME

§ 5 - **En conclusion**, il semble clair que l'outil informatique s'avère puissant au niveau didactique pour contrôler et activer certaines variables dont on mesure mal l'effet dans les cadres traditionnels de l'expression orale ou écrite de la classe. Il permet, en atténuant l'influence de l'affect dans la relation de l'élève au savoir, de faire émerger des procédures spontanées et naturelles et embrassant des populations de taille importante (sans imposer un plan d'expérience lourd et complexe), d'analyser des régularités dans ces procédures . Ainsi , l'émission de conjectures sur le plan des stratégies d'ingéniérie didactique trouve un fondement moins empirique que celui qu'un enseignant peut formuler au vu des productions des élèves de sa classe. Aussi, nous continuerons dans la voie dialectique d'une part, de perfectionnement, efficacité et accessibilité de logiciels, d'autre part, émission, opérationalisation et évaluation d'hypothèses didactiques. C'est, nous semble-t-il, à travers des telles dualités que l'intégration du micro-ordinateur dans le processus d'enseignement puisera son séns et convaincra de son utilité.

ANNEXE

CLASSIFICATION DE 31 MODALITES DE REPONSE ET LES EFFECTIFS D'ELEVES CORRESPONDANTS


A
R1 1-3-5 (63)
R6 2-1-8 (70)
Q5 6,5-2-9 (6)
Q6 2-1-6 (4)
Q2 3-4-3 (2)
Q3 2-5-2 (4)
R2 3-4-6 (77)
R3 2-5-8 (70)
R5 6,5-2-7 (68)
redondance
13
15
17
18
14
B
Q1 1-3-1 (14)
Q3 2-5-4 (12)
Q5 6,5-2-11 (11)
Q4 3-6-3 (17)
Q4 3-6-10 (41)
6
9
11
5
Maintien des instanciations, changement des relations par rapport à l'attendu des théorèmes
Liaison Hyp.Concl. par les signes
21
C
Q1 1-3-2 (6)
Q1 1-3-9 (9)
Q2 3-4-5 (4)
Q3 2-5-6 (4)
Q3 2-5-7 (4)
Q4 3-6-9 (4)
Q6 2-1-3 (12)
Q6 2-1-4 (5)
Q4 3-6-5 (6)
Q5 6,5-2-3 (7)
changement des instanciations des variables changement des relations par rapport à l'attendu des théorèmes
19
pas de liaison
10
7
20
D
Q1 1-3-3 (10)
Q1 1-3-6 (4)
Q2 3-4-2 (4)
Q4 3-6-4 (4)
Q6 2-1-5 (6)
changement des instanciations ; maintient du sens du théorème dans le choix de la conclusion
Liaison Hyp.Conc par le sens
8
2
16
E
Q2 3-4-10 (3)
R4 3-6-6 (58)
12

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GENDER DIFFERENCES IN MATHEMATICS LEARNING REVISITED(1)

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Abstract

Over the last decade much attention has been focused by practitioners, administrators, school systems, governments, as well as researchers, on gender differences in mathematics learning. In this paper data gathered through interviews in 1988 with students in grade three and six classes are discussed. Particular emphasis is placed on affective variables believed to impact on mathematics learning. It is argued that summary statistics need to be supplemented with interview data for an accurate and comprehensive description of continuing gender differences in mathematics learning.

The National Policy for the Education of Girls (Commonwealth Scools Commission, 1987) formalized quite explicitly the commitment of the Australian government to gender equity in education. The proclaimed policy summarised and built on practices and initiatives already in place in various States and systems. Its recommendations included

> the development of educational programs and related action in primary and secondary schools ... which will enable all Australian girls and boys to develop their potential.

Well before its publication, special funds had been made available for programs mounted to redress the disadvantages faced by girls in traditional education settings. Though not its main purpose, the study described here allowed an examination of the impact of such initiatives on factors associated with mathematics learning. While the work, which is still in progress, is concerned with both cognitive and affective factors, only information relating to the latter is presented here.

AIMS

The main aim of the study was to monitor students' processing of teachers' explanations. The participating teachers were informed that this was the purpose which motivated the research. However, the data collection procedure used also allowed a number of gender differences to be examined.

METHOD

Mathematics lessons were videotaped at various times during the school year. Selected excerpts were replayed to the students in a one-to-one setting to explore what they heard and processed when teachers explained and discussed mathematical concepts. Analysis of these data are still in progress.

Various strategies were used to facilitate rapport between the students and the interviewer during these sessions. For example, before turning to the videotapes students were asked questions about their background, general interests, and reactions to different aspects of mathematics lessons. These sessions were either videotaped or audiotaped. Inspection of the students' responses revealed a sufficient number of continuing gender differences to warrant closer examination of the data.

Measures of student achievement were collected on two occasions: during terms 2 and 4,i.e., one quarter of the way through and towards the end of the school year respectively.

The sample

The sample comprised 94 students, 43 girls and 51 boys, in grades 3 and 6 in four different schools in the same region in Melbourne, Australia. The schools were identified by a senior regional officer for their interest in research and their progressive educational philosophies.

Instruments

The main data gathering instrument, i.e., for the data reported here, consisted of structured interviews. Since their nature and scope become apparent in the reporting of the results, they are not described in more detail here.

Two measures of student achievement were obtained: through the administration of mathematics tests and through teacher rankings. The Progressive Achievement Tests in Mathematics (Australian Council for Educational Research, 1984) was administered to the grade three students; The Class Achievement Tests in Mathematics (Australian Council for Educational Research, 1976) to the grade 6 students.

Prior to the administration of the tests teachers were asked to rank their students into three groups: above average, average, and below average.

RESULTS

No significant differences were found in girls' and boys' overall performance on the mathematics tests. This held for each class, at each grade level, and for both testings. The broad overlap between teacher rating and student score attained on the tests can be summarised by reporting that in each class students rated above average had the highest mean score, followed by those rated as average, with the students rated as below average having the lowest mean scores. However, a number of subtle differences were noted in teachers' ratings of their students. Despite the very similar performances of the girls and boys on the tests, collectively teachers rated 25% of the boys but only 14% of the girls in the above average group, and 18% of the boys compared with 28% of the girls in the below average group. When asked a comparable question about their own performance, 25% of the boys and 14% of the girls indicated that they considered themselves to be above average; 10% of the boys and 28% of the girls below average. Responses to a number of other questions are summarised below. Because of the open ended nature of the data gathering technique, more than one option was offered by some students to a number of the questions asked.

•Spare time activities:

While there was much overlap in girls' and boys' leisure time activities, girls seemed to spend more time on sedentary indoor activities; boys on active outdoor pursuits.

•Favourite lesson:

Mathematics was nominated by 51% of the boys and 21% of the girls; sport by 18% of the boys and 9% of the girls; language by 14% of the boys and 42% of the girls.

•Difficulty of mathematics

Thirty three per cent of the girls considered mathematics to be easy, compared with 12% of the boys; 26% of the girls and 45% of the boys believed mathematics to be difficult; the rest 'couldn't say' or believed that 'it depends on the work'.

When faced with a difficult mathematics problem, about two-thirds of both the girls and the boys would ask the teacher for help. Strategies of trying again or returning to the problem later were volunteered by about one-third of the boys and approximately half the girls. Surprisingly, very few students indicated that they would ask a friend for help.

•Strategy for catching up, after an absence

Friends featured strongly in the replies to this question. About two-thirds of the boys and half the girls indicated that they would ask a friend first. Often specific individuals, typically of the same sex, were nominated. About a quarter of the boys and 40% of the girls would first turn to the teacher. The remainder were evenly divided between 'mum', 'dad', and 'don't know'.

•Doing mathematics alone or with a partner

Despite some strong individual preferences, most students indicated that they liked both.

*Perceived parents' attitudes to mathematics

Approximately 20% of the boys and 10% of the girls indicated that their parents had said that mathematics was an important subject and that they, the students, should try hard to do well. Some 40% of both sexes said that they had no idea what their parents thought about mathematics or that they believed them not to be particularly interested in it.

*Who is better at mathematics?

The majority of the students (75% of the boys and 60% of the girls) believed that girls and boys were equally good at mathematics. Nevertheless, 9% of the boys and 15% of the girls expected girls to do better; 15% of the boys and 8% of the girls boys to be better.

To supplement these representative summative data, a number of student responses are reported at some length.

Katrina

Katrina is a grade 3 student. She scored or shared the top mark in her class at both testings and is considered to be well above average by her teacher. She does not have much spare time because

> I usually go to gym and when I am at home I like to go up and play with the pets.

Mathematics is her favourite lesson at school. She thinks

> mathematics is fairly easy if you know what you are doing....If I listen I know what to do, if I don't, I don't know what to do, so I listen.

She thinks that she is near the top of the class and can think of nothing that she dislikes about mathematics. When she comes across a difficult problem

> I just try and wait to see if it's wrong. I try as hard as I can. (If it is still wrong) I try and see if I can work it out again because I was probably thinking of something else when I did it.

She likes doing mathematics on her own and has some hesitancy about working with a partner.

> Sometimes when I work with others they tell me. I don't feel quite as good because sometimes when they do that, they mis-tell me and I end up with the wrong sum, 'cause they mis-told me....(But) I don't mind helping others.

She thinks that girls are probably better at mathematics than boys. In fact

> all the girls in our grade are a lot better at mathematics than the boys. ... I think you should try. Some people do learn more than others. But it would be nice if we all learnt the same. Then people wouldn't tease you. In our grade they like teasing - mostly the boys do.

Her parents think that mathematics is a good subject to study and that it is a fairly easy subject to learn as well.

Simon

Simon is also in grade 3, in the same class as Katrina. He scored equal first and equal second on the two testings and is also perceived to be a good student by his teacher. He likes reading cricket magazines in his spare time. Sport is his favourite lesson at school. He really likes doing sums but gets frustrated when the teacher's instructions are not exact.

> Like, when she says we're going to do 12 sums, so I put down 12 and then we do only 10.

He prefers doing difficult sums

> especially something new that we haven't done before.

When he comes across a difficult problem he simply goes over to the teacher and tells her

> I don't understand this. And then she just explains it to me.

Generally he likes doing mathematics on his own. If he has been away for a few days he asks his teacher for some extra help. His parents think mathematics is 'good'. Boys and girls, he believes, are about equal when it comes to mathematics.

Steven

Steven is a grade 6 student who obtained the second and third highest scores in his class on the two testings. He is regarded as a good student by his teacher. In his spare time he likes

> to build things with lego. And my mum said to learn my tables so sometimes I do sums.

Mathematics is his favourite lesson at school.

> It's good. I like it. You need to learn it for when you go to a job. Everyone has to know mathematics.

When asked to describe a typical mathematics lesson, he said

> They're good. First we usually do some work together. Then we do the work on the board. You write in your book and work them out.

He particularly likes

> just sitting in the quiet. Figuring out things. Doing hard sums and that. I don't like it when we have real easy ones.

His preference is for doing mathematics on his own because

> it's fun and quiet. I like working in a group a bit. Well, you need to be cooperative.

If he'd been away he would ask Nathan, his best friend, for help. He believes that boys and girls are the same at maths:

> Like a boy could do some things better than a girl, but a girl could do other things better....My mum is real good at it. My dad needs it to do diameters and things for houses at work. But he's not that good. Sometimes he gets stuck.

Helen

Helen is in the same class as Steven. She scored the highest mark at both testings and is also described by her teacher as a good mathematics student. In her spare time she likes to type, do mathematics games on the computer or perhaps read. While she generally likes school

> I'd probably say mathematics is my favourite lesson because it's fun to do ... because I understand it and it's easy. Some people think it's hard but when you know what you are doing you enjoy it.

In a typical lesson

> Mr. N. goes through each question two or three times. Examples or whatever. And when he's finished that, some of the people in the class can ask questions. Then when that's all finished we can start on our own.

She considers herself to be one of the top at mathematics. When she encounters a difficult problem:

> Sometimes I ask Mr. N. and he gives me hints and I get it straight off. Sometimes I just sit there and wait until it comes into my head.

She can not say whether girls or boys are better at mathematics because

> it all depends on how smart you are, not whether you're a boy or girl.

Her parents

> are pretty strict on maths. They say I'm not to fool around in maths so I'll pass and all that. They want me to be pretty good in maths. My dad's good and sometimes he makes me these little stories and I have to work them out.

CONCLUDING COMMENTS

In many ways the data presented here merely confirmed those of earlier research. Despite the considerable number of programs mounted to promote equity in mathematics learning, girls and boys with comparable achievement in mathematics perceived themselves, and were perceived by their teachers, differently in a number of subtle ways.

The results were reported in two ways: through summative statistics which emphasized between-group differences and through more detailed interview extracts which facilitated recognition of individual and within-group differences. The former tend to reinforce popular stereotypes while the latter are not only more likely to serve as a challenge to them but at the same time point to the students most at risk. Collectively, the data reveal the continuing need to monitor affective components of mathematics learning.

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References

Australian Council for Educational Research (1976) *Class Achievement Tests in Mathematics* Hawthorn, Victoria: Author

---------- (1984) *Progressive Achievement Tests in Mathematics* Hawthorn, Victoria: Author

Commonwealth Schools Commission (1987) *The National Policy for the Education of Girls in Australian Schools* Canberra: Author

La résolution de problèmes dans l'enseignement des mathématiques: compte rendu d'une expérience auprès d'enseignants du primaire

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Abstract

Mathematical problem solving is a part of the curricula for primary and secondary school levels. Viewed by most of the teachers and the authors of the curricula as a royal mean to construct mathematical habilities and knowledge, problem solving heuristics constitute an object to teach. In order to get a better understanding of that phenomena, we conducted an experiment with teachers in which they were invited to solve problems they judjed difficults, to discuss about the heuristics they used and to clarify the rôle of mathematical problem solving in the teaching of mathematics.

1. Discutant des heuristiques, les enseignants découvrent que les mathématiques ont un contenu

Intéressés à infléchir les conceptions des enseignants sur les mérites d'un enseignement des heuristiques de résolution de problèmes mathématiques, nous les invitons à résoudre des problèmes et à discuter des heuristiques alors appliquées. De cette expérience, la majorité des enseignants ne retiennent que les connaissances mathématiques qu'ils ont pu construire. Le compte rendu et l'analyse de cette expérience visent une interprétation de ce résultat.

2. La résolution de problèmes dans l'élaboration des savoirs en mathématiques

L'activité des mathématiciens, productrice de savoirs savants, est déclenchée et modulée par les problèmes, les conjectures et les questions envisagés par ces chercheurs (Brousseau, 1986). La résolution de problèmes caractérise cette activité de construction de savoirs. Le mathématicien partage ainsi avec sa communauté scientifique, non seulement des savoirs mais également des savoir faire. L'intérêt des didacticiens et des enseignants pour ces savoir faire apparaît pleinement justifié.

3. La résolution de problèmes dans l'enseignement des mathématiques et la recherche en didactique des mathématiques

L'intérêt des enseignants pour la résolution de problèmes, a toujours été. Comme le fait remarquer Conne (1989), les problèmes sont des "archétypes dans l'épistémologie commune" des enseignants qui les proposent depuis des siècles aux élèves; cet intérêt trouve sa justification dans les astuces de raisonnement ou les heuristiques qui ennoblissent cette activité qui apparaît alors une manifestation éclatante de l'intelligence générale. La notion d'heuristique s'avère également des plus commodes dans l'interprétation des échecs ou des difficultés des élèves en mathématiques, au centre d'élaborations qui préservent les identités des enseignants et des élèves. Enfin, les enseignants ne peuvent généralement que recourir aux heuristiques pour rendre compte de la construction de leurs connaissances en mathématiques; ces heuristiques constituent également dans l'échange enseignant-élèves les entrées du modèle de construction de connaissances que l'enseignant entend transmettre aux élèves.

La résolution de problèmes est également un objet privilégié par les chercheurs en didactique des mathématiques. Les études réalisées depuis les dix dernières années (Kilpatrick, 1985; Kintsch & Greeno, 1985; Krutetskii, 1976; Mayer, 1983; Schoenfeld, 1985; Vergnaud, 1981, 1982) ont modifié les perceptions initiales sur les heuristiques de résolution de problèmes; peu de chercheurs en didactique des mathématiques ne songent maintenant à dissocier les heuristiques de résolution des contenus mathématiques des problèmes et ne croient utiles d'imposer aux élèves des démarches de résolution de problèmes.

Ces modifications des connaissances et des prescriptions des chercheurs en didactique des mathématiques n'ont pas encore atteint les enseignants et les curricula en mathématiques. Bien au contraire, on observe depuis un certain nombre d'années une inclusion d'heuristiques de résolution de problèmes mathématiques dans les programmes d'enseignement de cette matière. On assiste donc à la création d'un nouvel objet d'enseignement: les heuristiques , voire même, les algorithmes de résolution de problèmes. Ce glissement métadidactique relève-t-il d'une transposition didactique? Notre interprétation des études du phénomène de transposition didactique (Chevallard, 1985; Brousseau, 1986) nous incite à le penser.

4. Les objectifs de la présente étude

Les questions suivantes sont à l'origine de cette étude: a) Quelles sont les connaissances des enseignants sur les heuristiques de résolution de problèmes? b) Quel rôle les enseignants attribuent-ils à la résolution de problèmes dans l'enseignement et l'apprentissage des mathématiques? c) Quelles sont les heuristiques que ces enseignants mettent spontanément en oeuvre dans la résolution de problèmes mathématiques? Correspondent-elles à celles qu'ils préconisent dans leur enseignement? d) Quelles sont les heuristiques que ces enseignants mettent en oeuvre dans une activité de résolution de problèmes étalée sur plusieurs jours, exigeant de multiples tentatives de résolution de problèmes jugés complexes ou difficiles? e) Cette dernière activité de résolution de problèmes modifie-t-elle leurs perceptions de la place et du rôle de la résolution de problèmes dans l'enseignement des mathématiques?

5. La séquence didactique

36 enseignants (28 étudiants en formation des maîtres et 8 enseignants en perfectionnement) participent à cette expérience.

5.1. première étape

Les enseignants sont d'abord invités à répondre individuellement aux questions suivantes: a) Quelles sont les heuristiques (stratégies) de résolution de problèmes mathématiques que vous connaissez ou utilisez; parmi celles-ci, quelles sont selon vous les plus efficaces? b) Quel rôle attribuez-vous à la résolution de problèmes dans l'enseignement et l'apprentissage des mathématiques?

5.2. seconde étape

Une banque de problèmes leur est ensuite présentée. Les problèmes retenus sont puisés des problèmes discutés par Schoenfeld (1985) et Mayer (1983); une traduction et une adaptation sont effectuées. À titre d'exemple, le problème suivant: "Une distance de 363 km sépare deux villes. Jean et Paul décident de se rencontrer. Si Jean parcourt 1 km la première journée, 3 la seconde, 5 la troisième et ainsi de suite, et si Paul parcourt 2 km la première journée, 6 la seconde, 10 la troisième et ainsi de suite, quand se rencontreront-ils?"

Les enseignants disposent chacun de 2 heures pour résoudre ces problèmes. Ils sont invités à essayer de résoudre tous les problèmes et à indiquer pour chacun les heuristiques qu'ils utilisent et le temps de résolution. Leurs solutions sont ensuite examinées; les problèmes apparemment les plus difficiles sont retenus. Puis, chacun des enseignants se voit contraint de résoudre le problème qu'il juge le plus difficile; il dispose alors d'une période de 2 semaines; il enregistre sa démarche et rédige un rapport écrit de son activité.

Suit une présentation commentée des heuristiques de résolution de problèmes décrites par Schoenfeld (1985). Les analyses de chacun des enseignants sont alors discutées en fonction de cette présentation; des questions contrôlent cette discussion: Quelles sont les heuristiques dont fait état votre analyse? Quelles sont les connaissances qui président à l'évocation de l'une ou l'autre de ces heuristiques? Est-il possible d'ordonner ces heuristiques et de suggérer une séquence d'application de ces heuristiques? Existent-ils des heuristiques plus efficaces selon le contenus mathématiques?

5.3. troisième étape

Quatre équipes sont formées et doivent résoudre le problème suivant:

Un automobiliste part de Montréal et se rend à Québec; il effectue ce trajet à une vitesse moyenne v. À quelle vitesse doit-il revenir à Montréal, s'il veut que la vitesse moyenne pour tout le parcours (aller-retour) soit $2 v$?

Une heuristique différente est imposée à trois des équipes, la dernière équipe pouvant choisir les heuristiques qui lui semblent appropriées: équipe 1: représentation graphique; équipe 2: représentation numérique; équipe 3: représentation algébrique. Les solutions de chacune des équipes sont discutées.

5.4. quatrième étape

Les questions formulées à la première étape sont reprises à cette dernière étape; un examen des réponses est alors effectuée et les enseignants sont confrontés à leurs réponses initiales.

6. Examen des comportements

6.1. les perceptions initiales des enseignants

Invités à préciser le rôle de la résolution de problèmes dans l'apprentissage et l'enseignement des mathématiques, les enseignants ne font jamais appel à leurs activités en mathématiques. Ils invoquent trois points de vue essentiels: le point de vue de l'enseignant qui tente d'expliquer les réussites et les échecs des élèves en résolution de problèmes; le point de vue de l'enseignant qui relève certaines observations sur l'efficacité de certaines situations de résolution de problèmes, de certaines tâches, en regard des heuristiques dont elles peuvent susciter l'application; le point de vue de l'enseignant sur les processus de résolution de problèmes, relevant davantage de la psycho-pédagogie que de la didactique.

6.2. les heuristiques appliquées par les enseignants au cours du travail prolongé en résolution d'un problème

Les heuristiques de résolution de problèmes mises en oeuvre au cours de la première tentative de résolution des problèmes qu'ils jugent difficiles sont peu variées chez ces enseignants: lecture répétée des textes, inscription de certaines données, calculs immédiats ou encore, représentation algébrique de certaines données. Peu de schémas ou de dessins sont construits. Tous abandonnent ces problèmes après avoir procédé à certains calculs.

Ces comportements se modifient par la suite; contraints de poursuivre la résolution d'un de ces problèmes, plusieurs enseignants mettent en oeuvre diverses heuristiques et peuvent les évaluer en tenant compte des connaissances en jeu dans les problèmes.

Dans la discussion des protocoles, la majorité des enseignants peuvent reconnaître les heuristiques de résolution de problèmes décrites par Schoenfeld (1985). Ils constatent également que l'application d'heuristiques dépend de connaissances spécifiques; ils s'entendent aussi sur le fait qu'une heuristique n'assure pas la réussite d'un problème. S'ils nuancent alors leurs jugements initiaux sur les heuristiques, la discussion n'entraîne pas une modification des conceptions du rôle des heuristiques dans la résolution de problèmes et une dissociation des notions d'heuristiques et d'algorithmes.

6.3. L'application contrainte d'heuristiques particulières par les enseignants

Devant appliquer une heuristique spécifique pour résoudre le problème sur la vitesse (troisième étape), quelques enseignants seulement parviennent à résoudre ce problème; certains résolvent d'abord le problème à leur façon (une solution numérique, en général) puis produisent une solution adaptée à l'heuristique demandée.

6.4. le bilan réalisé par les enseignants à la suite des activités de résolution de problèmes

Le bilan des enseignants sur les résultats de l'expérience qu'ils ont vécue ne comporte presqu'exclusivement que des références aux connaissances mathématiques construites. Ainsi, placés dans une situation analogue à celle des élèves auxquels ils s'adressent, devant ainsi résoudre des problèmes non routiniers et appliquer certaines des heuristiques qu'ils suggèrent normalement aux élèves d'appliquer, ils sont amenés à s'interroger sur la pertinence de ces heuristiques et à lier ces heuristiques aux contenus mathématiques des problèmes et aux connaissances dont ils disposent. Cette activité leur permet avant tout de construire diverses connaissances mathématiques; pour cette raison, bien peu parmi eux affirment avoir également élaboré des savoir faire en mathématiques.

On peut faire l'hypothèse que ces constatations sur leurs expériences puissent les conduire à modifier leurs perceptions du rôle et de la place de la résolution de problèmes dans l'enseignement des mathématiques. Cette hypothèse serait, dans notre étude, non confirmée. En effet, lors de la reprise des questions initiales, la majorité des enseignants s'appuient sur les différences entre leurs situations d'enseignants et les situations d'élèves pour rappeler l'importance de proposer des démarches de résolution de problèmes aux élèves; les raisons invoquées sont de cette nature: contrairement aux adultes, les élèves ne savent pas comment aborder les problèmes, il faut donc leur enseigner; les élèves n'ont pas encore fait suffisamment d'activités mathématiques pour être en mesure d'identifier les stratégies pertinentes, il convient donc de leur faire découvrir l'importance de ces stratégies à travers des activités variées de résolution de problèmes. Les enseignants concluent en déclarant que cette expérience leur a permis toutefois de découvrir que les contenus mathématiques des problèmes doivent être examinés avant de proposer des heuristiques, un tel examen leur permettant de proposer des heuristiques plus pertinentes.

7. Références

Brousseau, G. (1986). Théorisation des phénomènes d'enseignement des mathématiques, Thèse d'état, Université de Bordeaux 1.

Chevallard, Y. (1985). La transposition didactique, Editions La pensée sauvage, Grenoble.

Conne, F., Pauli, L. (1989). Initiation à une réflexion sur le rôle du langage dans l'enseignement des mathématiques, *Petit x*. (à paraître).

Kilpatrick, J. (1985). A retrospective account of the past twenty-five years of research on teaching mathematical problem solving, in E. A. Silver (Ed.).*Teaching and learning mathematical problem solving*. Hillsdale, N. J.: Erlbaum, 1-15.

Kintsch, W., Greeno, J. G. (1985). Understanding and solving word arithmetic problems, *Psychological Review*, 92, 109-129.

Krutetskii, V.A. (1976). *The Psychology of Mathematical Abilities in School Children*. University of Chicago Press.

Mayer, R. E. (1983). *Thinking, Problem Solving, Cognition*. Freeman: New York.

Schoenfeld, A. H. (1985). *Mathematical Problem Solving*. Academic Press: New York.

Vergnaud, G. (1981). Quelques orientations théoriques et méthodologiques des recherches françaises en didactique des mathématiques, *Actes du 5 e Colloque PME, 2*, 7-17, Grenoble.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems, in T. P. Carpenter, J. M. Moser & T. A. Romberg (eds.). *Addition and subtraction: a cognitive perspective*, Hillsdale, N. J.: Erlbaum.

STRATEGIES USED BY 'ADDERS' IN PROPORTIONAL REASONING

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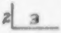
Adders are students who consistently used the incorrect-addition strategy on some hard ratio items. English adders used addition-based methods on most of ratio items. Both written test papers and interview data showed that Taiwan adders used predominantly the taught multiplicative algorithms on easy ratio items. The strategies Taiwan adders used and the reasons they made their errors were examined in the interviews.

The Incorrect Addition Strategy And Adders

When asked to enlarge  so that the new base line is 5 units, a child concentrating on the difference 5-3 rather than 5/3 will say: "5 is 2 more than 3, so the new upright is 2 more than 2, answer 4 units". Such a strategy for solving proportional items is called the incorrect-addition strategy.

In both Hart's (1981) CSMS ratio study and its replicated study in Taiwan (Lin et al, 1985), the incorrect-addition strategy occured most frequently on four 'hard' items (the addition-type questions), namely, Mr. short question (the missing value paper clips task), enlargement with ratio 3:5 of a -shape and enlargement with ratios 8:12 and 12:8 of a -shape. In both cases, a significant feature of the performance of children operating at the lower levels of understanding was their use of the incorrect-addition strategy (Hart, 1981; Lin et al, 1985).

Theoretically, Piaget and Inhelder (1958) describe the incorrect addition strategy as a typical answer from a child at the late concrete stage. However Karplus et al. (1975) and Hart (1984) see it not as an inevitable consequence of developmental level but as a method which should be corrected. In order to develop appropriate diagnostic teaching procedures, it is necessary to investigate why children make this kind of error. Consequently, Hart (1981,1984) investigated the prevalence and context of this strategy within English sample in detail.

Hart (1981) described those students who consistently used the incorrect-addition strategy to solve at least three out of the above four addition-type items as 'adders'. There are about 30% (resp.20%) of adders in English (resp. Taiwan) children population of aged 13-15. It was found that adders are not in any particular age group. Most of adders are not the least able. .

According to Hart's (1984) findings, English adders used the additive methods consistently on ratio problems. Where the relationship between the values involved was simple (such as double, half, three times ... etc.), the students were able to use their additive methods to obtain a correct solution. Difficulties arose when the numerical relationship were more complex. In these cases, the 'adders' were not able to apply their methods correctly. Instead, they resorted to a simple addition of the given values, This given rise to the 'incorrect-addition strategy'. In general, English adders' approach was characterised by

a) using addition-based child-methods, such as halving, doubling, adding on and building up to solve easy ratio items;

b) avoiding applying multiplication of fractions, and taught algorithms; and

c) never using multiplicative strategies, such as the unitary method (how much for one), and the formula method (a/b = c/d).

Instead of the taught multiplicative methods, English adders used prevalently their own child-methods on ratio tasks. In Taiwan, Lin (1988) showed that the taught algorithms is the only system in junior high school mathematics. With such difference, it is a matter of interest to examine whether the strategies used by Taiwan adders were similar to English adders or not. This paper therefore sets out to study the characteristics of Taiwan adders.

Methodology

A sample of 33 adders, aged 13-14, were identified from six classes in three typical schools in Taiwan.

Each of these adders was interviewed for about 30-40 minutes during the same week that the written test was given. On interview, each one was shown the test paper completed earlier, and asked to explain the answers to some items they had completed. Since the question under examination was whether Taiwanese adders' performance is characterised by the same features as the CSMS adders' performance, students were interviewed on selected 'easier' items where correct answers could be found by additive methods, as CSMS adders did, as well as on the addition-type questions. This was necessary in order to determine if the items which they had answered correctly had also been handled by additive methods.

Following students' explanations and bearing in mind the features identifies for the CSMS adders, the interview focussed on investigating the following questions:

a) Do 'adders' ever use multiplicative strategies?

b) How did adders decide to use either an additive or a multiplicative strategy?

c) What kind of understanding of fractions do 'adders' have ?

In order to provide a fuller description of adders' characteristics, the following three aspects which were not covered in Hart's (1981, 1984) studies were also examined:

d) Are adders aware of non-integer multiples?

e) Are adders aware of two kinds of ratio, ratio of two portions either within one figure or between two figures?

f) How good is adders' recognition ability for distinguishing non-ratio contexts from ratio contexts?

Findings

1. Methods used by Taiwan adders

Besides the four addition-type questions, fourteen easier items in the test paper were used in the interview. Out of 33 adders, twelve adders consistently used multiplicative algorithms before they faced the four addition-type questions. Six adders used the correct multiplicative strategies consistently. Ten adders used approximately the same number of additive and multiplicative strategies on these easy items. Four adders used additive strategies on ten or more easy items. Only one adder never used a multiplicative strategy on the easy items.

When the numerical relation involved were easy (e.g.8:4; 5:10; 5:15) Taiwan adders were very often able to apply the taught algorithms correctly. Most of them concentrated on either 'how many for one' or 'the multiple relation of the values involved' and used the unitary method and the multiplier method. 9 adders used correct additive strategies, such as repeated addition or adding on the extra units according to the multiple, on the eel and its food items with ratio 5:10 and 5:15.

On a recipe item where the amounts for 8 people was given and the amounts for 6 peoples was asked, 8 out of 33 adders viewed 6 people as the sum of 4 people and 2 people and used the building up strategy to solve it.

When the numerical relationship were more complex (e.g. 8:6; 10:15; 15:25), about half or more of adders faced their difficulties. Some adders were doing 'undirected manipulation'. They manipulated the given data in one or two procedures, similar to the correct multiplicative procedures. However there appeared to be little idea of how to reason proportionally. Some adders were using 'bigger / smaller so then multiplying / dividing' strategy. They felt the need to operate multiplicatively. However, the multiple / divisor used is not the enlargement factor, but is very often small integers. Some adders were using 'bigger / smaller so then adding / subtracting' strategy. They felt that some extra units were needed to add / subtract for a bigger / smaller one. The number of extra units can usually be identified from the nearby context, but sometimes can be arbitrary.

On the four addition-type items, 8 out of 33*4 responses were different from the incorrect-addition strategy. Five of them used correct multiplicative methods and the other three were incorrect responses. On the geometric enlargement questions, Hart (1984,p.23) says "the children's answers ... provided a clear indication that multiplication was not used to produce an enlargement. The children stated that the only way of obtaining 12 from 8 was by addition." So, for English adders, these items seemed to be 'natural' addition-type questions. Many Taiwan adders were aware that multiplicative is appropriate for these items. However, due to some individual reasons, they chose the incorrect-addition strategy.

2. Reasons for switching to the incorrect-addition strategy

(i) Non-awareness of non-integer multiple

27 out of 33 adders on the interviews said that 'there is no multiple relation between 5 and 3'. Non-awareness of a non-integer multiple as a multiple, was the main reason that most of the Taiwan adders switched their multiplicative algorithms to the incorrect-addition strategy.

(ii) The geometric settings

In relation to some studies, Karplus et al. (1983) concluded that "the occurrence of dimensions inhibits the (incorrect) addition strategy". The four addition-type questions were concerned with enlargement of figures and with non-integer ratios which were dimensionless. Apart from the complexity of the numerical relationship, the geometeic settings also create obstacles, as some excerpts from the interviews showed.

> "...same shape just means the length of a bigger diagram is increased, so I add..."
>
> "...straight, can use ratio. Curved, I am not sure if I can use ratio. In a flash, I saw the relation of constant difference, I decided to use it."

(iii) Affective reasons

> "...all items were using multiple relation before (on this test); the test paper might have a trick, it is impossible that each item was 'multiple'. I had such an experience before. Since and look different, I changed my method."

This adder interpreted his distraction in terms of his belief about test papers which had developed because of his previous testing experiences.

In Taiwan, very often students are trained to solve problems as quickly as possible. Because subtraction was quicker and easier,so adders chose it.

> "...during the examination, I tried to choose between subtraction and multiplication. Subtraction was easy. I have checked the answer, both with constant difference of 4 units (the -shape item). It was right, so I chose subtraction."

This adder even evaluated and felt happy about his choice, for he had been reasoning 'logically'.

3. Other findings

(i) Poor understanding of fractions

14 out of 33 adders used fractions on the test paper. Only six of them could use fractions to amplify their multiplicative algorithms. In the interviews, about 2/3 of adders could do computation of fractions by taught algorithm. However, most of them could not apply it appropriately in any context, as their test papers showed. They have a poor understanding of fraction.

(ii) Awareness of 'Within' and 'Between' ratios

Regarding the geometric enlargement questions, the between ratio method is used by the adder who concentrated on the ratio of two corresponding portions between two figures; the within ratio method is used by the adder who concentrated on the ratio of two portions within a figure. Most of the Taiwan adders were aware of one kind of ratio but might not have been aware of both kind of ratios. None of the adders in the interviews showed the ability to chose the more economic one among two kind of ratios.

(iii) Distinguishing ratio from non-ratio contexts

Some Taiwan adders identified the type of question, additive or multiplicative,by checking some key words in the problem sentences. Some, when they used the native methods, 'bigger so adding/multiplying' and 'smaller so subtracting/dividing', chose their operations in terms of how comfortable they felt about the numbers to be operated on.

In order to investigate their process of solving problems, a non-multiplicative task with surface structure similar to missing value proportional item was asked during the interviews. 14 out of 33 adders used the multiplier method to solve it. They tended to solve the problem by repeating methods used on the ratio test paper.

4. Summary

In terms of the findings, some characteristics of Taiwan adders could be summarized as follows:

a). Using multiplicative algorithms predominantly on easy ratio items.

b). Thinking of multiplying, however their multiplicative methods are not secure on harder items with ratios 2:3, 2:5 etc.

c). Switching to the incorrect-addition strategy because of, e.g. non-awareness of non-integer multiple, obstacles of the geometric settings, on the addition-type questions.

d). Manipulating fractions by taught algorithms; poor understanding of fractions.

e). The process of solving problems were inappropriate, very often were divorced from understanding.

f). Non-awareness of non-integer multiple.

g). Only aware of one kind of ratio, within or between ratio.

h). Poor ability of distinguishing contexts.

Discussion

Almost all the Taiwan adders used taught multiplicative algorithms on the easy items of the ratio test. Unlike the English adders who are only working in the addition system, Taiwan adders are also working in the multiplication system, especially on easy items. From this point of view, Taiwan adders are not really 'adders', in the true sense of that word.

Evidence from other Taiwan studies (Lin, 1988) Suggests that the main reasons for these differences lie with the very different teaching approaches to which the two sets of students are exposed. In Taiwan the teaching emphasis is on conventional algorithms while in the UK, the teachers encourage students to make use of whatever method suits them best so students develop what Booth (1981) called 'child-methods', "which are based on counting, adding-on or building-up approach, and by which children attempt to solve mathematical problems within a human-sense framework".

Implication

In terms of the findings in this study, some suggestions should therefore be made for developing diagnostic teaching modules which we hope to be of benefit to all students.

1. Taiwan adders were able to use multiplier and unitary methods on easy ratio problems. Therefore, instruction which is based on the 'for-every' strategy (Case, 1978; Gold,1978) could link to their abilities and therefore be of benefit to them.
2. Activities which lead to 'cognitive conflict', as Hart's (1984) module emphasized, have proved to be very motivating in bringing about cognitive and conceptual change (Kuo et al.,1986). Apart from these activities it would be better for every lesson to include some non-ratio questions so that adders have the chance to distinguish the difference between a constant ratio relationship from a constant difference relationship. This activity was also suggested by Karplus et al. (1983). In such a way, they can gradually develop an appropriate process of solving problems, i.e. based on understanding.
3. In terms of poor understanding of fractions, the concept of a fraction and its operations should be emphasized. Using a calculator to grasp the

meaning of 7/3 and 3/7, as Hart's (1984) module showed, has proved to be very effective for Taiwan adders (Kuo et al., 1986).

4. A formal method of finding multiples, i.e. find x in a•x=b , should be learned and the awareness of non-integer multilpe should be developed.
5. Both within and between ratios should be emphasized so that the flexibility of choosing the more economic kind of ratio to match the context can be developed.

REFERENCES

Booth, L.R. (1981): Child-Methods in Secondary Mathematics, Educational Strdies in Mathematics, 12, 29-41.

Case,R. (1978): A developmentally based theory and technology of instruction. Review of Educational Research, 48.

Gold,A.P. (1978): Cumulative learning versus cognitive development. Ph.D. thesis, University of California, Berkeley.

Hart, K.M. (1981): Children's Understanding of Mathematics:11-16; John Murray.

Hart, K.M. (1984): Ratio:Childern's Strategies and Errors. Windsor: NFER-Nelson Publishing Company Ltd.

Karplus, R., Karplus, E., Formisano, M. and Paulson, A.C. (1975): Proportional Reasoning and Control of Variables in Seven Countries. Advancing Education Through Science Programs, Report ID-65.

Karplus, R., Pulos, S. and Stage, E.K. (1983): 'Early Adolescents' Proportional Reasoning on 'Rate' Problems', Educational Studies In Mathematics, 14,219-233.

Kuo, F.P., Lin, F.L. and Lin, K.H. (1986), Diagnostic Teaching: Ratio, A report of the Mathematics Understanding of Taiwan students (MUT) programme, Dept of Mathematics, National Taiwan Normal University (in Chinese).

Lin, F.L., Kuo, F.P. and Lin K.H. (1985): Junior high school children's understanding of ratio and proportion, a second year report of the Mathematics Understanding of Taiwan students programme (MUT), Dept. of Mathematics, Taiwan Normal University (in Chinese).

Lin, F.L. (1988): 'Societal Differences and Their Influences on Children's Mathematics Understanding'. Educational Studies in Mathematics 19, v.4.

Piaget, J. and Inhelder, B. (1958): The Growth of Logical Thinking from Childhood to Adolescence. London: Routledge and Kegan Paul.

Canonical Representations of Fractions as Cognitive Obstacles in Elementary Teachers

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THE EXPERIENCE SUGGESTED TO STUDENTS IN THE CONTEXT OF FRACTIONS IS TOO RESTRICTED AND LACKS THE REQUIRED COMPLEXITY. IT IS BASED ON TWO OR THREE STEREOTYPES WHICH WE CALL CANONICAL. THESE STEREOTYPES LEAD TO A NARROW CONCEPTION AND CAN EASILY CAUSE MISCONCEPTIONS AND CONFUSIONS. IN A SAMPLE OF ELEMENTARY TEACHERS THAT WE EXAMINED WE ACTUALLY FOUND ALL THESE MISCONCEPTIONS AND CONFUSIONS.

Frege's characterization of whole numbers is not only an ingeniously mathematical achievement. It can also be considered as a deep psychological insight. It tells you, if you wish to interpret it this way, what the cognitive requirements needed for constructing the meaning of the whole numbers are. (According to Frege, the number five, for instance, is the class of all sets which contain exactly five elements).

Thus, many psychological claims made about the child conception of number can be considered as claims whithin Frege's arithmetical paradigm. Here you can count Piaget (1952) and many others as Gelman (1978), Skemp (1971) and Steffe et al (1983). We can illustrate this point by the following quotation:

Skemp (1971, p. 144-146) asks: WHAT DO WE MEAN BY "THREE"? His answer is: "THREE" IS THE CHARACTERISTIC PROPERTY OF A CERTAIN COLLECTION OF SETS OF WHICH WE CAN CHOOSE A SUFFICIENT VARIETY TO ENABLE OUR STUDENTS TO FORM THE CONCEPT ITSELF.

When dealing with fractions, one sees immediately that this domain has striking inferiority relative to the domain of natural numbers. We do not refer by this to the well known fact that fractions are harder for the students than whole numbers. What is really missing is (1) a mathematical definition which is also psychologically valid and, as a result of it, a characterization of the concrete experience required in order to acquire the abstract concept of fraction (analogous to the concrete experience implied by Frege's definition). We would like to suggest a definition of a fraction which immitates Frege's definition of a whole number. This definition, so we hope, contains also the psychological elements of the fraction concept and thus has the potential of suggesting concrete experience required in order to acquire the fraction concept.

DEFINITION: a fraction m/n, $n\neq 0$, $0\leq m\leq n$, **IS THE CLASS OF ALL TRIPLETS** in the first place of which there is a whole, in the second place there is a partition of the whole into n equal parts and in the third place there are m parts of the partition.

Note that by this definition we have defined only proper fractions and we have not defined the notion of rational number. Namely, we have not defined the equivalence of fractions. But one can easily see how to define the missing concepts by applying the above definition. Also the above definition lacks an additional condition on the whole which we omitted in order to avoid complications. However, the implicit assumption there is that some measure is associated with the whole. It can be length, area, volume, weight, etc. in case of a fraction of a continuous quantity and can be the number of elements in case of a discrete quantity. In this paper we will deal with fractions of continuous quantities.

Note that the notion of the whole is essential in our definition. It implies that it has to be clear of what whole a partition is going to be made. Our impression is that this problem is ignored by most methods of teaching fractions and this fact leads to many well known confusions and misconceptions of students and teachers. In addition to that, the concrete experience suggested to students in relation with the fraction concept is by no means not rich and not variegated as the concrete experience they get in relation with the whole number.

When introducing fractions as **CONTINUOUS QUANTITIES** there are some stereotypes which we call canonical that block the way to the abstraction required in order to acquire the fraction concept according to our definition. The most common whole with which students interact when learning fractions is the circle.

Sometimes they see also squares or rectangles which are not squares. As a result of using the circle as a whole, the partition into equal parts becomes a partition into congruent parts. This causes sometimes a failure to identify fractions in case where the parts are equal but not congruent. The fact that the question of the whole is not discussed explicitly and that implicit assumptions are very often involved causes sometimes confusion.

Examples and test items relating to this confusion one can find in Peck and Jencks (1981), Hart (1979, p. 66) and Lesh et al (1983, pp. 309 - 336).

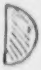
The goal of our study here was to examine elementary teachers conceptions about the points raised above. The research questions were:

1. To what extent elementary teachers are flexible when the canonical whole is replaced by another whole?
2. To what extent do they realize that the awareness to the question what the whole is determines sometimes the success on fraction tasks?
3. To what extent do they realize that the partition of the whole does not have to be to congruent parts?
4. To what extent do elementary teachers have non-canonical representations for fractions?

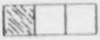
Method

Several interviews with elementary teachers were made and as a result of these the following questionnaire was formed:

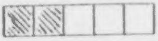
1. What is the whole if the following figure is 2/3 of it?



2. Students were asked to evaluate 1/3 + 2/5. One student drew:

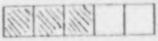


1/3



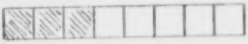
2/5

and got the answer:



3/5

Another student drew similar representations for 1/3 and 2/5 and got the answer:



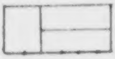
3/8

Is the first student correct? Is the second student correct? Please, explain!

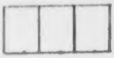
(This question is based on Peck and Jencks (1981)).

3. Please determine in each figure whether it was divided into thirds. If there is a mistake in a figure, please, explain it!

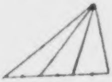
(a)



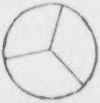
(b)



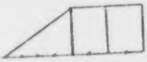
(c)



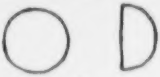
(d)



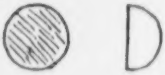
(e)



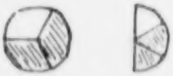
4. How will you illustrate to a student the meaning of 2/5. Please, do it in at least two different ways!
5. A teacher asked her students to mark 2/3 of the following configuration.



One student drew



Another student drew



Is the first answer correct? Is the second answer correct? Please, explain!

The reader can easily see that questions 1 and 5 in the questionnaire were designed to answer research question 1. Question 2 in the questionnaire was designed to answer research question 2, question 3 in the questionnaire was designed to answer research question 3 and question 4 in the questionnaire was designed to answer research question 4.

The above questionnaire was distributed to 237 teachers and 72 pre-service teachers. 54 teachers out of the 237 had the official title of Mathematics coordinators in their schools. These are teachers who have more interest in mathematics than the average teacher and also underwent some in-service mathematical training.
In the result section they will be referred to as math coordinators while the other teachers will be referred to as teachers.

Results

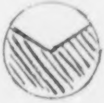
Question 1 was designed to examine directly whether the canonical representation of the fraction as a part of a complete circle is an obstacle in the way to the correct answer in a non-canonical situation. The results are given in Table 1.

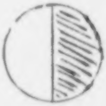
TABLE 1	The whole is the complete circle	The whole is 3/4 of the circle	I do not know or it is impossible to know
Pre - service teachers (N = 72)	43%	34%	24%
Teachers (N = 183)	29%	47%	23%
Math coordinators (N = 54)	21%	7%	72%

Note that the first column indicates the expected misconception. The complete circle is the whole, no matter what additional information is given. The second column indicates a correct but possibly narrow conception, since they are infinitely many ways to complete the given figure in order to obtain a whole.

This was expressed by one of the math coordinators who claimed that "it is impossible to tell what the whole is. We can only evaluate its area. There are infinitely many wholes 2/3 of which is the given figure". This can explain the fact that 72% of the math coordinators are in the third column. The percentage of the incorrect answer in the math coordinators was the least in the three subgroups but it was also the least in the case of the correct but possibly narrow conception of the fraction. Since verbal explanations were missing in most of the answers we cannot tell whether somebody is in the third column because of a correct or an incorrect reason.

The most common drawings for the claim that the complete circle is the whole were:





In the first one, the figure which was given in the question was ignored. In the second one, the arithmetical information which was given in the question was ignored. Such phenomena occur when stereotypes are so dominant that they attract all the attention and the additional information is ignored.

As to question 5 we believe that a teacher who has a flexible conception of the ways to represent fractions would have claimed that both drawings are legitimate and correct. But only 42% of the entire sample demonstrated such flexibility. The details are in table 2.

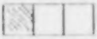
TABLE 2	The rigid conception (only 1 drawing correct)	The flexible conception, both drawings correct	Other
Pre - service teachers (N = 72)	62%	32%	6%
Teachers (N = 183)	46%	44%	10%
Math coordinators (N = 54)	44%	52%	4%

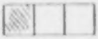
Question 3 included three (out of five) non-canonical representations of thirds. In order to claim that (a), (c) and (e) are partitions into thirds a certain geometrical knowledge is required and we were not sure that all the teachers in our sample had it. What we assumed was that all of them had the geometrical knowledge required for (a). Hence, teachers who claimed that only (b) and (d) were partitions into thirds were considered by us as people who believe that the parts of a partition representing a fraction should be congruent. We believe that if these teachers were aware of the fact that the parts of a partition can be equal without being congruent they would have examined (a) and would have come to the conclusion that it is a partition into thirds.

TABLE 3	All the partitions, partitions into thirds	Only (b) & (d) are partitions into thirds	Only (a, b & d) are partitions into thirds
Pre - service teachers (N = 72)	20%	51%	26%
Teachers (N = 183)	34%	38%	25%
Math coordinators (N = 54)	52%	15%	33%

Note that the common canonical representation prevented them from examining (a) and their failure in (a) is due to the lack of a conceptual understanding and not to the lack of geometrical knowledge.

As we pointed in our introduction, it is extremely important on given fraction tasks to be aware of the question what the whole is. This point is not emphasized enough in textbooks or by teachers. In order to illustrate (1/3)+(1/3) many authors use:





without mentioning explicitly what the whole should be.

This leads immediately to the mistakes presented to our teachers by question 2. We were interested to see what percentage of the teachers could explain conceptually the childrens' mistakes. An answer like "the child is wrong because (1/3)+(2/5) are 11/15" is not considered as a conceptual explanation. Of course, it is better than an answer justifying one of the childrens' results (and unfortunately there were some answers like that). Nevertheless, such an answer is not satisfactory because it does not have any conceptual explanatory power. It indicates probably that the teacher does not have a conceptual understanding of the situation. An indication of a conceptual understanding can be the claim that the whole should remain the same through the entire process of adding.

TABLE 4	A claim of wrong answers with conceptual explanation	A claim of wrong answers with NO conceptual explanation	A claim that one of the answers is correct
Pre - service teachers (N = 72)	14%	66%	20%
Teachers (N = 183)	16%	66%	18%
Math coordinators (N = 54)	26%	61%	13%

Together with Table 4 we should consider also the answers to question 4. Here, only 2 preservice teachers out of the entire population gave non-canonical representations for 2/5. This is in spite of the fact that the question asked for at least two different representations. This is not surprising because in construction tasks usually the dominant representation is evoked in the mind and thus the respondents could use a circle and a rectangle or two non-similar rectangles and to divide them into congruent parts. Therefore, in order to get a more accurate picture, one should consult Table 3 which relates to the identification task. This table, when taking a liberal criterion to which columns 1 and 3 are the columns of the correct answers, shows us that at most 61% of the entire sample realize that non-canonical representations are legitimate representations of fractions.

Discussion

The above results show that teachers' visual representations of fractions are incomplete and unsatisfactory. They are not sufficient to form a complete concept of fraction. We do not intend here to recommend specific learning aids which can improve the situation. The direction is clear. One should provide the student with various non-similar representations. We are aware of the risk of various representations. They might confuse the student. Therefore, it is worthwhile in this context to quote Behr et al (1983, p. 124): "Contrary to the prevailing opinion among Mathematics educators, we have learned that a "good" manipulative aid is one that causes a certain ammount of confusion. The resultant cognitive disequilibrium leads to greater learning".

REFERENCES

Behr, J.B., Lesh, R., Thomas, R.P., and Silver, E.A.,
1983 "Rational-Number Concepts" in Acquisition of Mathematics Concepts and Processes, edited by R. Lesh, and M. Landau, New-York: Academic Press.
Gelman, R., and Gallistel, C.R.,
1978 The Child's Understanding of Number. Cambridge: Harvard University Press.
Hart, K., ed.,
1981 Childrens' Understanding of Mathematics: 11-16, Murray.
Lesh, R., Landau, M., and Hamilton, E.,
1983 "Conceptual Models and Problem-Solving Research". In Acquisition of Mathematics Concepts and Processes, edited by R. Lesh and M. Landau, New-York: Academic Press.
Peck, D., and Jencks, S.,
1981 "Conceptual Issues in the Teaching and Learning of Fractions" in Journal for Research in Mathematics Education, Vol. 12, No. 5, pp. 339-348.
Piaget, J.,
1952 The Child's Conception of Number, London: Routledge and Kegan Paul.
Skemp, R.R.,
1973 The Psychology of Learning Mathematics, Middlesex, England: Penguin Books.
Steffe, L.P., Glasersfeld, E., Richards, J., and Cobb, P.,
1983 Childrens' Counting Types, New-York: Praeger Publishers.

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USING CONCEPT MAPS TO EXPLORE STUDENTS' UNDERSTANDING IN GEOMETRY

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Concept maps were used before and after a teaching program on the topic of parallel lines. The maps were studied to identify which concept names were familiar to the students, and which propositions the students were able to construct. The concept maps revealed some misconceptions that were not evident in other forms of testing that were also used.

Concept maps have been used extensively in some subjects, notably science, as a method of studying students' knowledge and understanding of various topics. According to Novak and Gowin (1984), a concept map "is a schematic device for representing a set of concept meanings embedded in a framework of propositions" (p.15). As an evaluation tool, concept maps can be used to determine what concepts are familiar to students and what links the students have formed between the concepts.

A constructivist view of learning holds that students construct knowledge in the context of actions on objects, including ideas, and reflections on those actions. New knowledge might occur as the addition of new information to the structures already held by the student, or alternatively a more radical restructuring of the student's existing knowledge may be made to accommodate new information. For meaningful learning to take place, students must choose to relate new information to relevant concepts and propositions they already know. For this reason, it is necessary for the teacher to try to establish the main ideas and relationships that each student has at the beginning of a new unit of work. The teaching must then be designed to challenge the views the student already has, and to compare and contrast the student's views with those of other students and the teacher. Individual interviews are one appropriate means of probing students' views and their use in studies of teaching and learning is well established. However, their use in larger scale studies or by teachers in daily classroom activities is not always viable.

In our research on students' misconceptions in geometry, we have used individual interviews in earlier phases of our study where the number of students was relatively small. In the latest phase of our study, however, we had twelve teachers in ten different schools using our teaching materials on parallel lines. We decided to ask the students in these classes to construct a concept map concerning parallel lines as part of written pre- and post-tests. Following is a report on the concept maps constructed by one class, and what we learned from them.

Procedure

The class of 29 Year 8 students (12 girls, 17 boys) attended a high school in a disadvantaged suburb of Perth, Western Australia. The tests were administered before and after the teaching program by their mathematics teacher, who also taught the program.

The tests consisted of three parts: (1) the construction of a concept map, (2) a set of propositions and (3) a set of drawings. In part (2) the students were asked to indicate which of fourteen propositions about parallel lines they thought were true, false, or were unsure about. In part (3), they were asked to indicate whether each of the ten drawings showed lines that were parallel or not parallel. Part (1) first showed a simple concept map made by a Year 8 student to show the link between some ideas about fractions. A statement pointed out that the ideas were written inside the boxes in the map and that the lines with arrows showed the links between the ideas. Ten ideas related to parallel lines were then listed. The students were asked to draw a map showing the links between the given ideas, to label the links, and to add other ideas about parallel lines if they wished.

These students had not received instruction previously on how to construct a concept map. Other researchers (Edwards and Fraser, 1983; Novak and Gowin, 1984) have stressed the importance of introducing students carefully to the techniques of constructing concept maps but physical constraints prevented us from doing this. These students were, therefore, confronted with a novel task that not only required them to construct and organize propositions about a topic, but also to interpret the given example and to understand what the task was asking of them. We expected that these students would find the construction of a concept map very challenging.

Intrinsically, constructing concept maps is a more difficult task than the other tasks in our tests, since to construct a concept map students have to work out a hierarchy of the given concepts, construct propositions to link those concepts, and make an intelligible spatial arrangement of the concepts. In part (2) of our tests, the propositions were already constructed and the students had only to decide between three responses. In part (3) of our test, no propositional thinking was necessarily involved.

To evaluate concept maps, Novak and Gowin (1984, p.36) suggest a numerical score with different weightings given to the number of meaningful, valid propositions shown, the number of valid levels of the hierarchical arrangement of the concepts, valid and significant cross links between sets of related concepts or propositions, and examples given of concepts. Other authors such as Brumby (1983) have used coding and scoring procedures that cannot be transferred readily to other contexts. Our purpose was to see what propositions about parallel lines the students were able to construct before and after instruction, which concept names were familiar to them, and whether they had any misconceptions that we were unable to detect in the other parts of our tests. Accordingly, we were interested in the number of propositions the students made and whether or not these were correct. An incorrect relationship suggests a misconception that the student has. Misconceptions can be localized and specific and exist in an otherwise satisfactory conceptual framework. Similarly, concept maps can identify omissions in a student's understanding. Since we provided the students with the ten concepts we wanted them to link, the omission of any of these from a map might suggest either that the student did not understand the concept, was not familiar with the word naming the concept, or could not name a relationship between that concept and others that were listed.

Findings

Of the 29 students in the class, 26 were present for both tests. In the pre-test, one student re-drew the given fractions concept map and added explanatory notes to it. Nine students did not attempt the concept map at all in the pre-test. As suggested earlier, this may have been because the task was unfamiliar, or because they did not understand the given concepts, or because they could not organize the concepts in any meaningful way. In the post-test, all the students constructed a concept map, even though there had been no instruction about them during the teaching program.

The numbers of correct, incorrect, and meaningless propositions made by the students were tallied. The data for the 16 students who completed both maps showed that in the post-test most students (10 out of 16) constructed more propositions, including both incorrect and meaningless propositions; four made the same number and two made fewer propositions. Twelve students made more correct propositions in the post-test, two made the same number (both zero), and two made fewer. If the number of incorrect propositions is counted, the students generally either made the same number of incorrect propositions (6 students) or slightly more incorrect propositions (7 students). Overall, the total number of propositions made in the post test was greater than in the pre-test (122 compared with 76) and the percentage that were incorrect and meaningless dropped (29.9% compared with 48.7%). these results suggest that in the post-test, the students were able and willing to make more propositions, and indeed correct propositions, while still showing some misconceptions.

The concept names that were omitted most frequently in the pre-test were "coplanar" (13 times) and "equidistant" (12 times). Both these words are not commonly met by students in everyday speech. While all concepts were omitted less frequently in the post-test, these two concepts were still omitted the most frequently (both 6 times). Both concepts were the focus of parts of the teaching program. Nevertheless, from the maps it seems clear that both remained difficult ideas for some students.

The difficulty the students had with "coplanar" was not surprising to us. In previous phases of our study we had observed students' difficulties with this concept. The difficulty with the concept "equidistant" was, however, a complete surprise, although we knew of the difficulty some students have in measuring accurately. An examination of the responses in part (2) of our pre-test showed that the propositions that caused most difficulty were "parallel lines have to be coplanar" and "parallel lines can be curved". In the pre-test, only three students said that parallel lines must be coplanar, while 21 were unsure. The students' responses indicated in the pre-test that they were unfamiliar with the concept "coplanar". In the post-test, only three were still unsure, while 18 answered "true" and 8 answered "false". The students now seemed to be familiar with the concept while not necessarily correct in recognizing that parallel lines must be coplanar.

The proposition "parallel lines have to be the same distance apart" was correctly answered by 22 students on the pre-test and 28 on the post-test. Note that in this proposition we did not use the word "equidistant". Evidently the word "equidistant" itself, although used in the teaching program, presented difficulties to some students. An examination of the links made between "equidistant" and other concepts in the students' concept maps suggested that some students thought that "equidistant" was a synonym for "equal length". We were only able to identify this confusion of language by an examination of the concept maps, since the proposition concerning equal distance apart caused problems to so few students.

The value of concept maps in showing misconceptions where the responses to propositions or interview questions might not was evident in several individual cases. For example, on his post-test map Nathan T made the link "parallel lines must be coplanar". However, he also made the links "intersecting lines are not coplanar" and "curved lines cannot be coplanar". These propositions suggest an incomplete understanding of the concept "coplanar" and identify what aspects of the concept need further exploration. Another concept that seemed to be difficult for some students was "alongside". For example, Graham clearly thought that parallel lines must be alongside (aligned rather than non-aligned segments). He stated "to be parallel they must be alongside" and "they can be on any angle as long as there (sic) alongside each other." Similar propositions were made by several other students. Sharlene made the comment "It's not necessary to be alongside each other but it does help to tell if they are parallel lines", perhaps summarizing the view that some students have about why they think being alongside is an important aspect of being parallel.

Finally, the overall style or appearance of the concept maps seemed to give an indication of how the students were able to organize the knowledge they had about parallel lines. Generally, the post-test maps were more complex than the pre-test maps, because more concepts were included and more links were made and labelled to show propositions. A few students made linear maps by linking the concept names together to form a sentence. For example, both of Anita's maps were linear, although she labelled the links in the post-test map and showed much greater understanding of the topic. She was not able to label cross links or to arrange the concepts hierarchically. An excellent example of a map that was arranged hierarchically was Gordon's post-test map. He organized his map to show three branches: things that "don't occur in parallel lines", things that "aren't very important for parallel", and things that "have to be" for parallel lines.

Natasha's pre-test map showed a different approach. She linked six of the concepts to "parallel lines" by illustrating them pictorially. She drew a vertical line, a curved line, equal lines, and slanting lines to illustrate the meaning of these concepts, without in fact making any valid propositions about parallel lines. Unfortunately, she was absent from the post-test so a comparison between her maps was not possible. Generally, despite the greater complexity of the post-test maps, there were some stylistic similarities between the students' pre-test and post-test maps, suggesting perhaps that the ways in which they interpreted the task and their ways of organizing the concepts had not changed substantially.

Conclusions

We acknowledge that the task of constructing a concept map was a new and difficult one for these students. Nevertheless, we considered that the task was worthwhile from our point of view. We were able to find out which concepts may have been unknown to the students both before and after the teaching program by looking at what concepts were omitted. We could not have obtained this information from part (2) of our tests which could have been answered by guessing. We were able to look at individuals' maps and identify some of their specific misconceptions which did not show up in either part (2) or part (3) of our tests. We identified a problem with language that was surprising to us. Finally, some students seemed to have a particular style that they used in constructing their maps which may suggest how they viewed the task they were set and the ways in which they were able to organize the concepts.

Used along with the other types of questions we have used, the concept maps added considerably to our understanding of what students learned about parallel lines and how they structured their knowledge of the topic.

References

Brumby, M. (1983). Concept mapping: structure or process? Research in Science Education, 13, 9-17.

Edwards, J. & Fraser, K. (1983). Concept maps as reflections of conceptual understanding. Research in Science Education, 13, 19-26.

Novak, J.D. & Gowin, D.B. (1984). Learning how to learn. Cambridge: University Press.

MENTAL IMAGES: SOME PROBLEMS RELATED TO THE DEVELOPMENT OF SOLIDS

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abstract
In the reference frame of the theory of Piaget a study on the complex role of mental images is presented.
Starting from problems related to the development of solids a plan of interviews was set up and group of pupils at different ages were observed. the main results of this research are discussed. The aim is to propose didactic suggestions not only to improve pupils performances in the particular task but also in the elaboration of mental images.

Introduction

The problem of the contribution of mental images to our thinking is certainly fascinating and till now has not been completely clarified. It is well known that very often our thinking is supported by images and this is particularly true in the case of mathematical thoughts. Thus a study of mental images turns out to be very important not only from the general point of view of exploring the process of origin and utilization of mental images, but also from the point of view of mathematical education, with the aim of identifying specific didactic variables related to the problem.

The reference frame we choose the Piaget's theory as our reference frame, but with the ambitious objective of clarifying and exploring more thoroughly the question. Piaget devoted much work to the problem of mental images, particularly in his book L'image mentale chez l'enfant [1] ; from Piaget's ideas we took the following main hypothesis as a starting point.

- the mental image is not an extension of perception, but it is an "interiorised imitation" ("imitation interiorisée"), so that the image has a great autonomy from the perceptive process.

- The structural schematizing character of mental images corresponds to that of imitation: that is to say that the image organizes the information following the request of a symbolic representation; the schematized aspect of an image is not so stressed as for a concept, but works in the same direction: so there is a relation between thinking and image transformations.

- Even if there is an evolution from a first level, when images are strongly affected by the incapacity of mastering the inversion of a transformation, to a second level when the images acquire a greater dynamism by means of the influence of the operations, certain static characteristics of the first level last.

The Hypothesis

The specific hypotheses of our research project arise from the choice of a particular mathematical problem. This is not a problem generally considered to be very important, but we find it very stimulating: it is the problem of the development of a solid. In Italy it is not a basic subject in the mathematical curriculum, even if primary school teachers always deal with it. Problems concerning nets of regular polyhedra can be found in nearly all the textbooks, mainly with a practical aim (for instance "how to construct a cube with a sheet of paper"), and it is in this way that it is treated by teachers, who do not give great importance to this kind of problem. On the other hand, besides the specific works of Piaget [2], there are not many studies devoted to this problem [4]. The basic aim of our research project is to obtain more information on:

- the influence of mental operations in the process of organization of mental images during the period of concrete operations;
- the static versus dynamic character of mental images.

A specific hypothesis is considered:

- there are two levels of complexity when one considers problems connected with the manipulation of mental images:
  - a first level when primary intuitions [3] are sufficient; the image is global, it is not necessary to coordinate intermediate processes to solve the problem;
  - a second level when the primary intuitions are not sufficient any more: an operative organization of images is required to coordinate them according to the composition of transformation.

The method

Since the aim of the research is mainly explorative, the interview method was chosen in order to provide the opportunity to observe attentively the behavior of each individual child. On the other hand a plan of the interview was set up which was always followed in the same way, repeating the same questions, giving the same explanations or suggestions. This allows a standard in the final collection of results.

Subjects

Two different age levels were chosen:

- 10-11 year old pupils, corresponding to the end of the primary school

12- 13 year old pupils, corresponding to the second and third grade of the secondary school

Materials

During the interviews the following were proposed to the subjects :

- models of solids: a cube, a regular tetrahedron, a prism with a base in the form of an equilateral triangle;
- sheets of paper where the nets of the same solids had been drawn.

The questions

The question set was organized following three different stages:

I - Showing each object one asks the name of the solid. After hiding it one asks the child to count the number of faces, vertices and edges of the solid.

II - After a very short explanation one asks the child to draw the net of the solid considered in the first stage. When the first drawing is done the child is asked if it is possible to do an alternative drawing, solving the same problem. Each stage provides a first moment when the question is put without the object available (after having shown it the solid is hidden) and a second moment, if the child does not succeed, when the the object is given to him. Finally each child is asked to verify his solving procedure using the object itself.

III - Successively the drawing of each solids is presented with the question : "Is this the net of a certain solid?" . If the child gives an affirmative answer one asks him to imagine the reconstruction and to color in the same color the segments on the perimeter corresponding to the same edge on the solid. According to the hypothesis of two levels of complexity there are two types of net:

Type A : following the straight strategy (as a "flower") fig. 1.

Type B: following a ("rolling") strategy where the composition of more transformations is involved fig.2.

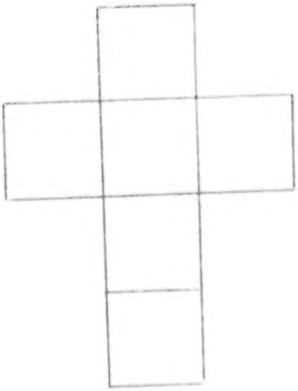

fig. 1

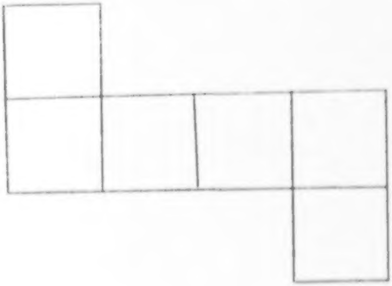

fig.2

Results

As regards the first step it is possible to establish a development in the systematic way of counting the elements of the solids (faces, vertices and edges). Roughly three different levels can be identified: absence of a systematic way, presence of a certain systematic method and a good systematic solution.

It is interesting to remark that at the intermediate level, if the object is not available, a rational organization overwhelms the mental representation of the object and counting reveals instabilities: for instance: the child counts the vertices grouping them by faces ("4 for each faces") and then multiplies by the number of faces, without realizing that some vertices have been counted twice. On the other hand at this level the presence of the object and particularly the handling of it often causes a failure: for example this is the case of Sara (10 years): even if without the object she has counted correctly, when she wants to verify her procedure using the object she counts turning the cube in her hands without any trace of order and fails. Generally it is possible to correlate a good systematic way of counting with a good performance in the development questions, but on the other hand often it is possible to find in the drawing of the net strategies related to the handling activity.

It seems possible to reinforce the hypothesis that the mental images supply a scheme useful for counting. Further it is clear that verbal language plays a basic role: the majority of the children improve their performances when asked to describe their counting strategies verbally. But this is a very interesting problem which deserves further specific study.

For the second stage there was the objective of verifying the hypothesis about the different kinds of intuition involved in the solution of the problem. Contructing the correct net of the solid implies coordination of a comprehensive mental representation of the object with the analysis of the single components (fa[illegible] vertices and edges). The results clearly show the presence of two different levels of complexity: the most commonly used strategy is that which corresponds, in our classification, to that of type A (a "flower"); in this case only primary intuitions are involved and it is not necessary to coordinate more than one transformation. As a further confirmation there is the fact that many children deny the possibility of the existence of other different nets of the solid. Naturally there is an improvement in this sense with age; but even with older subject many cases of failure are found so that it is possible to suggest the further hypothesis that without a specific stimulus there is no further evolution of the capacities in this field.

The role of mental operations is clearly shown by the results of the third stage of the interviews. As regards the question about the reconstruction it is found that the great majority of children interviewed succeed in recognizing the type A net, while they fail in the case of the type B net and consistent with their opinion in the previous stage, they even deny that the drawing proposed can be the net of a solid.

The description of the strategy provided by the children shows that the difficulty arises from the following fact: to correctly imagine the correspondence between the single segments of the perimeter reveals many more difficulties in the case of type B figures because in the process of reconstruction each element (faces, vertices ...) is transformed many times successively. To solve the task it is necessary to follow, in one's own mind , the transformations of the single element, so that the number of transformations represents an index of difficulty. As a result it is interesting to observe that the presence of a symmetry in the situation is not always noticed and used; mainly the youngest children do not even understand the suggestion regarding this possibility.

Conclusions

Thus, as we can see, it is possible to suggest a criterion to construct a hierarchy of difficulties in the task regarding the development of solids, based on a very general criterion related to the elaboration of mental images; on the other hand, following the same criterion, there are possible didactic suggestions useful not only to improve pupils' performances in problems related to the development of solids, but also to better organize didactics which have the more general aim of improving pupils' capacities in the elaboration of mental images.

References:

[1] J.Piaget - B.Inhelder, L'image mentale chez l'enfant. Paris, P.U.F., 1966

[2] J. Piaget, La représentation de l'espace chez l'enfant. Paris, P.U.F., 1947

[3] E.Fischbein, Intuition in science and mathematics. Dordrect, Reidel, 1988

[4] R.D.Bourgeois, Third graders'ability to associate foldout shapes with polyhedra, in Journal for research in Mathematical Education , Vol. 17 n 3 (1986)

[5] M.Pellerey, Procedimenti matematici e immagini mentali, in Orientamenti pedagogici 1984, 3

[6] U.Neisser, Cognition and reality, San Francisco, W.H.Wheeler & Co., 1976

[7] M.A.Mariotti - M. Sainati - M.Sciolis, L'immagine mentale per la formazione dei concetti geometrici, in L'insegnamento della Matematica e delle Scienze Integrate, Vol.10, n 5 (1987).

The Role of the Figure in Students' Concepts of Geometric Proof

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410 university students enrolled in a lower-division "core" mathematics course designed for nonscience majors were asked to judge the correctness of a proof of a geometric statement, and to assess the effect of using a different figure on the proof's validity. Two major findings emerged. First, use of non-generic figures did not appear to influence their judgments of the correctness of the proof; moreover, use of the special features of a non-generic figure did not appear to influence their judgments of the proof. Second, for many students, the proof appeared to be particular to the given figure; they indicated that a new proof would be required if a different figure were used. The "fit" between the two figures appeared to be a critical issue in determining whether the same proof could be used.

The concept of proof is one of the most important ideas in mathematics, yet research has shown that only the very ablest students achieve understanding of it (Senk, 1985; Williams, 1980). In a previous study, we found that many students do not limit their concept of mathematical proof to deductive arguments, but also accept inductive evidence as mathematical proof (Martin and Harel, 1989). Fischbein and Kedem (1982) showed that high-school students do not understand that statements mathematically proved to be true require no further empirical verification. Results of Vinner (1983) support this result and add the suggestion that high school students view a general proof as a method to examine and to verify a particular case — the *process* of the proof is generalized rather than the result of the proof; this also agreed with findings in our previous study (Martin and Harel, 1989)

In this study we address a further aspect of students' understanding of proof as establishing a result versus proof as establishing a process. A formal proof of a general statement usually involves various symbols and, in geometry a figure, which are used as external representations of the variable elements represented within the statement. Our question is whether students understand that these external representations do not influence the generality of the proof. For example, in an algebraic context, if a theorem is proved for three variables labeled x, y, and z, it is equally proved if they are labeled a, b, and c. In this paper we focus on this phenomenon in the context of geometry — to what degree do students of mathematics realize that the proof of a general geometric statement is not dependent on the particular figure accompanying the proof? More specifically,

1. Do students of mathematics realize that the proof of a general geometric statement is not dependent on the figure accompanying the proof? Conversely, do they realize that the proof of a general geometric statement may not depend on special features of the figure?
2. Do students conceptualize a geometric proof as a process that must be recapitulated in terms of a particular figure rather than as a proof of the statement for all figures?

Procedure

Instrumentation

Three parallel paper-and-pencil instruments were designed to aid in answering our research questions. In each instrument, subjects were presented with three situations, each presented on separate page of the instrument, related to the statement: *"A segment joining the midpoints of two sides of a triangle is* $^{1}/_{2}$ *the length of the third side."*

In the first situation of each instrument, subjects were presented with an argument and an accompanying figure purporting to prove the statement. They were presented with a fixed-response question about its correctness—"Yes, it is a correct proof" or "No, it is not a correct proof"—and asked to explain their response. In the second and third situations of each instrument, subjects were provided with an alternative figure (along with the original figure) and asked to evaluate whether the same proof would still work, in a fixed-response question—"The previous proof will work here", "We will need a new proof", or "I would need to look at the previous proof to answer the question"—and asked to explain their response.

Figures presented in the instrument varied in the degree to which they represented a generic triangle without special features. The proofs presented in the initial situation of each instrument differed in whether they were a general proof, or whether they relied on special features of the figure. The conditions of the three instruments are summarized in Table 1.

Table 1. Conditions found in Instruments 1, 2, and 3.

	Situation 1		Situation 2	Situation 3
	Kind of Proof	Kind of Figure	Kind of Figure	Kind of Figure
Instrument 1	General	General	General	Particular
Instrument 2	General	Particular	Particular	General
Instrument 3	Particular	Particular	Particular	General

Subjects and Method

One of the three instruments was presented to each of 410 students enrolled in a lower-level "core" mathematics course designed for nonscience majors at a large midwestern university; completion of a high-school level geometry course is a prerequisite for the course. The instrument was administered during a required class meeting in the twelfth week of the fifteen-week course. Subjects were allowed at least twenty minutes to complete their instrument; all subjects were easily able to finish.

Results

Responses to each of the instruments were analyzed using two different methodologies. First, frequencies of responses to the forced-answer questions were tabulated; see Table 2.

Table 2. Percentages of responses to forced-choice questions in Instruments 1, 2, and 3

	Response			
	n	Yes	No	Look again
Instrument 1				
Situation 1	130	87	13	(n.a.)
Situation 2	113[1]	78	15	7
Situation 3	113[1]	61	33	6
Instrument 2				
Situation 1	145	82	17	(n.a.)
Situation 2	119[1]	85	8	8
Situation 3	119[1]	30	65	4
Instrument 3				
Situation 1	135	85	15	(n.a.)
Situation 2	104[1]	61	31	9
Situation 3	104[1]	24	70	6

[1] Limited to students who responded "Yes" to the first situation.

Several observations can be made at this level of analysis:

1. Roughly equal ($\chi^2=1.224$; $p>0.20$) percentages of subjects accepted the initial proof in the first and second instruments. Thus, presenting the general proof with a non-generic figure

did not appear to influence subjects' judgment of the proof, relative to presenting the same proof with a more generic figure.

2. Roughly equal (χ^2=0.494; p>0.500) percentages of subjects accepted the initial proof in the second and third instruments. Thus, subjects did not appear to distinguish between a proof which is general but attached to a non-generic figure, and a proof which uses features of that same non-generic figure.

3. As seen in responses to each of the instruments, many subjects felt that the validity of a proof may be dependent on the figure used in explicating the proof. In each instrument, at least 15% of the subjects who had accepted the original proof were not convinced that the proof would work with the figure in Situation 2, which was "like" the original figure. In each instrument, at least 39% (ranging up to 76%) of the subjects who had accepted the original proof were not convinced that the proof would work with the figure in Situation 3, which was quite different in appearance from the original figure.

In attempting to further explain their beliefs of the role of the figure in a proof, subjects' explanations for their responses were reviewed and categorized. These were limited to Instruments 1 and 2 due to the inadequacy of the base-line task in Instrument 3.

The following major categories of response for subjects who felt that a proof would apply to the new figure were developed. Subjects categorized as General appealed to the generality of proof, as in the following response: "A proof that finds a statement should hold true for all examples of the same statement." Subjects categorized as Replay felt that the same proof could be applied or "replayed" in the current situation, as in "You could go through all of the steps you used before and get the right answer." Subjects categorized as Transfer focused on the new figure as a transformation of the previous figure, as in "All you did was enlarge the triangle and turn the triangle 180°." Subjects categorized as Statement focused on the statement rather than the proof, as in "If the statement is true, then it must work." The remaining students were categorized as No explanation or Other. See Table 3 for frequencies of categorizations.

Table 3. Frequencies[1] of categorizations for Subjects who accepted Situation 1.

	Instrument 1		Instrument 2	
Categorization	Situation 2	Situation 3	Situation 2	Situation 3
Accepted situation				
General	7	15	10	7
Replay	15	10	12	8
Transfer	56	12	42	3
Statement	1	10	10	8
Unclassified	2	11	12	0
No explanation	10	10	15	9
Did not accept situation				
Not Replayable	1	3	1	3
Not Transferable	12	32	5	70
Unclassified	5	1	3	4
No explanation	1	5	2	3

[1] Some respondents were classified in more than one way.

In the case of subjects who felt that a proof would *not* apply to the new figure, two major categorizations were developed. Subjects categorized as No-Replay felt that the same proof could not be applied or "replayed" in the current situation, as in "When the additional lines are added, the two corresponding triangles will no longer be congruent." Subjects categorized as

No-Transfer focused on the new figure as a "too different" transformation of the previous figure, as in "This is a new figure which is completely different." The remaining students were categorized as No explanation or Other. Frequencies of categories are presented in Table 3.

Several observations may be made from a review of these categorizations.

1. Relatively few students appealed to the generality of proof when arguing that a previously-accepted proof will work with a new figure. Many more students relied on surface features of "likeness" of the figures, as seen in the Transfer and Replay categorizations.

2. This same reliance on the surface features of figures lead many students to believe that a new proof would be required with a different figure, as can be seen in the No-Transfer and No-Replay categorizations. This effect was heightened as the figure differed more substantially from the original figure, as can be seen in the increased frequencies for these categorizations in Situation 3.

Discussion and Conclusions

We can summarize our findings to our research questions as follows:

1. Use of non-generic figures does not appear to influence students' judgments of the correctness of a mathematical proof. This finding is tempered by the observation that use of the features of a non-generic figure is also not seen as a problem.

2. Many students appear to conceptualize a geometric proof as a process that must be recapitulated in terms of the particular figure addressed. This can be seen in the number of subjects who wanted a new proof when presented with a new figure—indeed, a large proportion of these students indicated that they were basing this judgment based on the "fit" between the two figures. Further, even students who did feel that the same proof would be

valid with a different figure tended to base their judgments on "fit"; relatively few appealed to the generality of proof.

Further evidence of this phenomenum was found in 9 subjects categorized Transfer and 7 subjects categorized No-Transfer in Instrument 2. In Instrument 2, the figures in the first two situations were "skewed" in opposite directions; these students mentioned that the proofs would need to be reformulated reversing the role of the labels of several of the points. This represents very direct evidence for proof-as-replay.

Based on the findings on this study, the role of the figure in a geometric proof clearly requires additional attention, both in the instructional process and in future research.

References

Fischbein, E., & Kedem, I. (1982). Proof and certitude in the development of mathematical thinking. In A. Vermandel (Ed.), *Proceedings of the Sixth International Conference for the Psychology of Mathematics Education* (pp. 128-131). Antwerp: PME.

Henderson, K.B., and Smith, E.P. (1959). Proof. In *The growth of mathematical ideas grades K-12* (24th Yearbook of the NCTM, pp. 111-181). Washington, DC: NCTM.

Martin, W. G. and Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education, 20*, 41-51.

Senk, S. L. (1985). How well do students write geometry proofs? *Mathematics Teacher, 78*, 448-456.

Vinner, S. (1983). The notion of proof: Some aspects of students' view at the senior high level. In R. Hershkowitz (Ed.), *Proceedings of the Seventh International Conference for the Psychology of Mathematics Education* (pp. 289-294). Rehovot, Israel: Weizman Institute of Science.

Williams, E. (1980). *An investigation of senior high school students' understanding of the nature of mathematical proof*. Unpublished doctoral dissertation, University of Alberta.

The Inner Teacher, The Didactic Tension, And Shifts of Attention

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Abstract: Hirabayashi and Shigematsu have written several articles (1986, 1987, 1988) exploring the hypothesis of the Inner Teacher. Briefly stated, this hypothesis is that students tend to pick up suggestions and expressions of advice which they hear from their teachers, so that when they are working, it is almost as if they can hear their teacher's advice. Here we report on an opportunity taken to probe the Inner Teacher hypothesis in the context of Open University students attending a week long intensive mathematics summerschool while taking their first university mathematics course. The study is used as a springboard to examine connections between the Inner Teacher hypothesis, the Didactic Contract/Tension (Brousseau 1984, Mason 1986), and Shifts in the structure and nature of Awareness (Mason & Davis 1988).

BACKGROUND of THE COURSE and STUDENTS

Students of the Open University have to be over 21 years of age but otherwise need have no other qualifications. They study at home (about 10 hours a week for 32 weeks is expected for one course, and six courses make a general degree) from printed texts, television programmes and audiotapes. The printed materials make particular use of such suggestions as *clarify what you know, sort out what you want, build a bridge between them*; *when you are stuck, specialise, then re-generalise*. There is also one week's work devoted solely to these processes and their role in both learning and doing mathematics (Mason 1985). As part of their studies all students attend a week-long summer school, choosing from one of three sites over a ten week period. The summer school involves investigative mathematical exploration, with specific suggestions as to how to go about it, as well as revision.

THE STUDY

Two cohorts of students were asked questions before and after their summer-school week. The students present for the first week were asked two questions while waiting for the opening lecture to begin. The first was

Pre 1.1 Think back to your days at school. Can you recall any slogans, questions or advice that your teachers used to talk to you about working on or learning mathematics?

For the second week, the question was changed very slightly, because we regretted the word slogan, which may have triggered a particular form of reply.

Pre 1.2 Think back to your days at school. What advice or suggestions from your teachers about working on or learning mathematics can you recall?

There were 170 students registered for each week, so about 160 were probably there. Note that some students offered more than one reply.

Advice recalled from school	**Week One**	**Week Two**
None	74 (50%)	106 (69%)
Some aspect of memorising trig ratios	24 (16%)	7 (4%)
Some aspect of practicing using examples or reading the question carefully	15 (10%)	31 (20%)
Miscellaneous*	15 (10%)	10 (6%)
Totals	**128** replies	**154** replies

* For example *Don't say can't, say I will tomorrow; pay attention; show all your working.*

The second question posed both weeks before the opening lecture, was about more recent advice recalled from their course. Again, the question was altered slightly in the second week.

Pre 2.1 What advice, slogans or suggestions about working on or learning mathematics from the course text or tutor stands out for you now?

Pre 2.2 Think back to doing the last assessment assignment. What advice if any from the course or tutor came into your head?

Advice recalled from course/tutor	**Week One**	**Week Two**
None	81 (55%)	77 (53%)
Process Vocabulary Know & Want	7 (5%)	3 (2%)
Process Vocabulary Specialise & Generalise, Conjecturing	13 (9%)	2 (1%)
Miscellanous*	30 (21%)	45 (31%)
Specific to OU Study	4 (3%)	17 (12%)
Mathematical Content	10 (7%)	0
Total Student responses	**145** replies	**144** replies

At the end of the week students were asked

Post 1 What advice or suggestions struck you particularly during the week?

During the week there are a number of mathematical songs and slogans connected with mathematical topics (most notably QDQ^{-1} connected with matrix diagonalisation), which are inescapable.

Advice recalled from the week	Week One	Week Two
QDQ[1] and other such slogans	10 (9%)	7 (6%)
Process vocabulary: Know and Want; clarify question	10 (9%)	9 (8%)
Process vocabulary Specialise & Generalise, Conjecturing	40 (35%)	25 (23%)
Miscellaneous*	28 (24%)	35 (32%)
General comments**	2	24 (22%)
None (including *too many*; *nothing salient*)	25 (22%)	9 (8%)
Total number of student replies	**115** replies	**109** replie

* Many of these are connected with some striking incident or remark, eg week one, tutor "I've got my lucky knickers on", quoted by two students. Advice ranges from *be methodical*, to *take a break, don't panic, listen to others, . . .*

** Comments on the value of particular sessions, and other more personal comments like *I feel I'm thinking more mathematically.*

METHODOLOGICAL REMARKS

We find it useful to distinguish three points in a spectrum of probes, ranging from the explicitly *directive*, through *prompted or cued*, to *spontaneous* utterances. Thus, *do you recall . . .?* is a highly directed question, whereas a prompt or cue of the form *what advice did I give your colleague a few minutes ago?*, is intended to trigger recall, and, at the other extreme, one can look for spontaneous utterances by students which signal awareness of particular advice. We favour the spontaneous (Davis & Mason 1987) as the only unambiguous indication of a person having begun to internalise advice, integrating it into the automatic functioning of the inner teacher. Of course spontaneous utterances are few and far between, and therefore difficult to use quantitatively. Thus in a study of this form it is necessary to resort to probes of various sorts, and with groups of 150 students, such probes have to be fairly directive in order to elicit any analysable response at all.

The ambiguity present in interpreting directed and probed responses, stems from an instance of the Didactic Tension, a term derived by Mason (1986) from Brousseau's Didactic Transposition (1984). The tension, which necessarily pervades any teaching incident, is that the more explicit and precise a teacher is about the behaviour sought (as evidence of learning), the more likely it is that pupils will exhibit the behaviour mechanically, rather than as a result of understanding. In the case of the Inner Teacher hypothesis, the more explicit the researcher is about the kind of response being sought, or the kind of experience being looked for (eg the spontaneous welling up inside of a strategy for dealing with a particular mathematical situation), the more likely the subject is to provide such a response, not spontaneously, but as a result of having been prompted. For this reason, the questions posed to the students in our study were carefully chosen to be prompting but not wholly directing. It would have been perfectly possible, for instance, to ask students what advice they recalled from the course about what to do when you get stuck, since they had had a whole week's work on this, together with the same advice at various times in the mathematical texts. But the results would have provided even less evidence about the growth of an inner teacher.

The aim of the study was to probe the Inner Teacher hypothesis. The word *probe* was used intentionally, because it is not possible to prove or disprove the hypothesis by a study of this type. However, it is possible to use such a study as a springboard for further refinement and for reflection on some key issues in mathematics education centred on the inner teacher and the didactic tension.

ANALYTIC REMARKS

Hyabashi and Shigematsu (1988) were able to classify pupil replies in terms of *explanation, question, indication, or evaluation*. The replies we received were all in the form of questions or indications, but we found the distinction hard to sustain, because tutor questions sometimes emerged in the indicative mood, transformed into advice for the self.

The pre-week replies are sufficiently different between the weeks to attract attention. It is possible that more of the students in the first week were teachers, or that the precise wording of the questions triggered different memories. If the latter is the case, it highlights the sensitivity of replies to minor changes in wording, and hence the difficulty of getting spontaneous rather than prompted expressions of what students are thinking.

The comparison between pre- and post- week replies in both weeks is heartening, even when bearing in mind that these are prompted responses after an intensive week. Long term, the effects are likely to wear off, and there is a world of difference between recalling advice when prompted, and using advice when you get stuck, or even integrating the advice into your automatic behaviour. Yet with these adults, we can assert that a significant number became aware of the existence of advice, whatever the psychology of integrating that advice into appropriate behaviour.

The replies suggest that the initial question may have triggered the classic experience of tunnel vision when asked an unexpected question. Nothing particularly resonated with students waiting eagerly, and sometimes uncertainly for the week of mathematics to begin. Of those who did think of something in week one, the main advice recalled from school is connected with memorising particular facts, especially trigonometry. Although as indicated there were a variety of replies, we had no less than ten different versions of the *sine is side over hypotenuse* sequence SOHCAHTOA, ranging from the ordinary (Silly Old Hens Cackle All Hours Through Old Age) to the adult (Sex On Holiday Can Always Help To Overcome Anxiety). Other mathematical topics were similarly represented mnemonically.

A more careful conclusion is that, when asked these questions in the circumstances of waiting for the first lecture to begin on the first evening of what is for most a new experience, namely a whole week of mathematical activity, what came to mind was school mnemonics.

It is tempting to hypothesise that novelty plays a significant role in supporting long-term memory retrieval, and furthermore, that there is a cultural transmission of the mechanism of using novel mnemonics to remember facts - *"that's how things stick in your mind"*. Perhaps this leads teachers to construct new mnemonics for their pupils, taking pleasure in the novelty of their own particular version, but perhaps losing sight of the main aim which is for students to have ready access to the meanings of the trig ratio-names. The SOCAHTOA sequence was presumably itself intended to be memorable. Yet the convoluted two step process of recalling a sentence to get the letters and then decoding the letters to get the trig ratios seems far more effort than becoming imbued with the three trig ratios directly. Is it helpful to recommend such expensive (in terms of mental energy) strategies? Is there not a good chance that memorising of the mnemonic actually blocks further integration of awareness of trig ratio-names as ideas, so that it is not possible to subordinate and automate the awarenesses because they have been labelled and stored in a mechanical-linguistic way?

By contrast, in week two the dominant feature was the need to practice in order to succeed at mathematics. These replies could be seen as a manifestation of the culturally dominant impression of mathematics as a series of techniques which have to be practiced until they become automatic, so that you get the right answers. Some summer-school activities are intended to challenge this view, and the replies to the post-week question bear this out.

THEORETICAL DISCUSSION

The study reaffirms the observation that some students will not only notice the use of process vocabulary and advice from tutors, but also remember it, and even recall it when prompted by a question such as the ones we used. In Davis & Mason (1987) we reported on a similar phenomenon by students studying the same course, but with a longer time frame and with only spontaneous utterances as feedback from students. Hirabashi and Shigematsu (1986, 1987, 1988) also report similar experiences. How might we account for this, (as well as for its not being universal), and what issues does it raise for researchers and teachers?

We suggest that the phenomenon described is an instance of subconscious training of behaviour. In Davis & Mason (1987) we argued that the technique of repeatedly using the same language pattern (eg *What do you want?*, *What do you know?* etc) is more likely to register with students, even subconsciously, than constantly using different language for the same thing. Furthermore it is more likely to register if it is associated with an incident in which the advice actually helped (this observation lies behind the discipline of noticing elaborated in Mason 1987). And finally, such language is more likely to be noticed *as* advice, if attention is drawn explicitly to it, or some other means is used to invoke a shift in the nature and structure of students' attention (Mason & Davis 1988).

We suggest that the picking up of mannerisms, strategies and advice as described by the notion of the inner teacher, arises from investment (conscious or unconscious) in the teacher. Thus we would predict that it is more likely to happen in pre-adolescents who are (generally) eager to please; in adults who want to learn and who are impressed by what tutors seem to know; and by those adolescents who find a suitable role model in a particular teacher. We note that there can be deliberate mimicking (which is subject to the Didactic Tension in terms of its value to the student, since what is picked up is the behaviour rather than the understanding which generates the behaviour). There can also be deliberate rejection. Thus there is no easily identifiable general cause and effect operating, so it is necessary to look more closely. We conjecture that the significant factors are investment/respect; deliberate & explicit use at moments when the comments help the student out of difficulty; and supportive use of a gradual movement from *directed* introduction of advice, through *prompted* recall of that advice, towards unprompted, relatively *spontaneous* use by students.

It could be argued that recall is never purely spontaneous if the teacher is present, or even if the student is working on material associated with the teacher, because the very presence of the teacher, or the clasroom context may trigger recall. We suggest that this is but one step on the way to the educating of the inner teacher. There are close connections here with the *shift of attention* which comes from resonance induced by context or comment (Mason & Davis 1988). Resonance is the mechanism by which any association comes to mind, whether by expert or novice. The whole point of teaching is presumably to help students to integrate useful behaviour and to obtain access to that behaviour when appropriate. Integration comes by subordinating it (Gattegno 1987) to more automatic functioning, thereby releasing attention for higher order activity, and in particular, monitoring of activity (Schoenfeld 1986; Mason, Burton & Stacey 1984). Automatising behaviour can come through practicing (as stressed particularly by students in week two) and by more efficient means (Gattegno 1987, Tahta 1988) and is an important aspect of learning. To develop a wide base of resonance, the teacher chooses moments to invoke directed or prompted responses which are judged to be likely to make significant sense to students, and through attention being drawn to the interventions (a shift of attention), a rich web of meaning is built by students. The richer the web, the more likely the associated advice is to surface in times of need, or in other words, the more likely is the Inner Teacher to be heard.

James (1917) uses the term *Acting As If* to describe intentional change in mood, perspective and attention. The un-intentional picking up of behaviour patterns has much of the flavour of *acting as if*, in the sense that the student *finds themself* mimicking teacher behaviour (ie acting as if they were the teacher). We all experience it, especially when we suddenly notice that we have picked up a new word, phrase or cliché - sometimes even against our wishes. We suddenly become aware that we are using a particular phrase, and we can almost even hear our 'source' saying it. The force of the Inner Teacher hypothesis lies, for us, precisely in the brief moment of experiencing the *other*, or source. For a brief instant it is possible for the student, in a sense, *to be* the revered teacher. As Hyabashi and Shigematsu (1988) put it, "the teacher becomes another self of the pupil, monitoring and evaluating the original self's activity".

BEHAVIOUR and AWARENESS

Education is as prone to fad and fashions as any other human endeavour, and mathematics education has its fair share. Of particular relevance to this study is the opinion, often more implicit than explicit in what people say, that teachers should not (note the moral imperative - a sure sign of dubious reasoning) engage in activities which may prompt students to automatise behaviour superficially. Phrases such as *teaching for understanding* are often heard in discussions amongst teachers of mathematics, with the implication that anything which contributes to *rote* learning is unhelpful if not dangerous or irresponsible.

Our study is a reminder that this issue is not nearly as cut and dried as it seems at first. It is part of human nature to integrate behaviour and to subordinate or automate it. Furthermore it is only natural for pupils to wish to minimise the attention and energy needed to invest in manifesting the behaviour which the teacher seeks. This is the force which energises the Didactic Transposition (Brousseau 1984).

We suggest that there is nothing wrong in itself with training of behaviour, indeed we go so far as to suggest that *Only behaviour is trainable*, (an adage which is the essence of part of the framework used in Griffin et al (1988), for preparing to teach any topic). But hand in hand with behaviour goes awareness, and the inspiring assertion of Gattegno that *Only awareness is educable* (Gattegno 1976, Mason 1987, and used in Griffin et al 1988). Despite the current theology that understanding precedes automaticity, we suggest that responses such as those reported in our study remind us that the training of behaviour and the educating of awareness go together. Neither specifically or necessarily precedes the other. As we subordinate certain functioning, attention can be freed to attend to more executive type of control, and our awareness of appropriateness, of the range of relevant contexts, can grow correspondingly. As we exercise a skill we begin to see more ramifications than were visible on first encounter. Familiarity breeds contempt (prompts become superficial jargon or cliché), when it involves a loss of richness and stimulation. When familiarity has a sense of exploration, of uncovering greater richness, of stimulation (prompts become more meaningful) familiarity can also breed respect. Contempt and respect impinge on the affective domain, the third aspect of our psyche which is not wisely omitted. It has a correlate adage, *Only emotion is harnessable*, which links affect with motivation and drive. This is the essence of the mechanism exploited in Mason, Burton & Stacey (1984) in the form of *emotional snapshots*, and which has been developed much further recently in the discipline of noticing (Mason 1987, Mason & Davis 1989, and Jaworski et al 1989).

VALIDITY and CONSEQUENCES

As with our previous work, validity of our study lies, for us, in the extent to which it resonates with experience, and to which it awakens awareness of issues which they might otherwise have overlooked. If it helps some people to think about the relationships between the educating of awareness and the training of behaviour, or if it provokes further consideration of the Inner Teacher hypothesis, then it will have served its purpose. There has not been room or time here to develop teacher strategies which make use of opportunities noticed (triggered by awareness of the Inner Teacher hypothesis) in classrooms while teaching or researching, which is where the best test of validity and effectiveness lies.

BIBLIOGRAPHY

Gattegno, Caleb 1987 *The Science Of Education*, Educational Solutions, New York.

Griffin, Peter et al 1988 *Prearing To Teach Angle*, Open University Milton Keynes

Hyabashi, I. and Shigematsu K. 1986 Metacognition: the role of the inner teacher, proceedings of PME X , London p165-170.

Hyabashi, I. and Shigematsu K. 1987 Metacognition: the role of the inner teacher (2), proceedings of PME XI , Montreal Vol II, p243-249.

Hyabashi, I. and Shigematsu K. 1988 Metacognition: the role of the inner teacher (3), proceedings of PME XII , Vezprém, Hungary p410-416.

James, William 1917 The Gospel Of Relaxation, in *Selected Papers On Philosophy*, Everyman edition, Dent, London.

Jaworski, Barbara 1989 *Investigating Your Own Teaching*, unit 5 of the Open University course ME236: *Using Mathematical Thinking*, Milton Keynes.

Mason J. 1987 The Epistemethodology of Noticing, invited adress to British Society for Research Into Learning of Maths, Sheffield, ed. S. Pirie.

Mason, John & Burton, Leone & Stacey, Kaye 1982 *Thinking Mathematically*, Addison Wesley, London.

Mason, John & Davis, Joy 1987 The Use of Explicitly Introduced Vocabulary in Helping Students to Learn, and Teachers to teach in Mathematics, PME XI ed. J. Bergeron, N. Herscovics & C. Kieran, Montreal Vol III p 275-281.

Mason, John & Davis, Joy 1988 Shifts Of Attention, PME XII proceedings ed. A. Borbás, Vezprém, Hungary p 487-494.

Mason, John & Davis, Joy 1989 A Radical Constructivist Epistemethodology, Journal of Structural Learning, to appear.

Mason, John 1985 *Learning and Doing Mathematics* Open University Course unit, revised edition 1989 published by Macmillan, Basingstoke.

Schoenfeld, Alan 1985 *Mathematical Problem Solving*, Academic press, London

Tahta, Dick 1988 The Science Of Education, *Mathematics Teaching* 125 p10-13.

LECTURE ET CONSTRUCTION DE DIAGRAMMES EN BATONS DANS LE PREMIER CYCLE DE L'ENSEIGNEMENT SECONDAIRE FRANCAIS

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Résumé

La présente recherche concerne le traitement des diagrammes en bâtons par les élèves du premier cycle de l'enseignement secondaire français. Les deux activités de lecture et de construction étudiées sont référées aux niveaux scolaires. Les résultats différencient la progression des performances relatives à chacune des activités et une interaction entre activité (lecture et construction) et objet (nombre et graphique).

Abstract

This research deals with the treatment of sticks-diagrams by the pupils of the first cycle of the french secondary education. The reading and construction activities are related to the school levels. The results show the improvement of the performances linked to each activity and an interaction between activity (reading or construction) and object (numbers or graphics).

Sur le plan le plus général, le recours aux représentations graphiques dans l'enseignement sollicite deux types d'activités : la lecture (qui conduit à l'interprétation) et la construction. Dans une première approche, les tâches qui renvoient à chacune de ces deux activités se distinguent par l'information disponible au depart. Dans une tâche impliquant l'activité de lecture, la totalité de l'information graphique est disponible d'emblée, il s'agit alors d'une tâche de réorganisation signifiante du materiel. En revanche, l'autre tâche impose la construction, le tracé, de l'information manquante. Par ailleurs, la relation entre données numériques et représentations graphiques autorise, sous certaines contraintes d'accessibilité(présence de la totalité des nombres dans un cas, ou possibilité de leur extraction à partir du graphique dans l'autre), l'extension des deux activités de lecture et de construction aux valeurs numériques des grandeurs représentées. C'est donc à une exploration des relations entre activité de lecture et activité de construction relatives à la trace graphique et aux nombres correspondants que le présent travail est consacré.

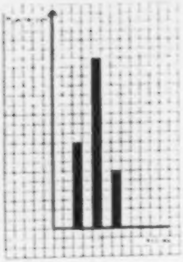
Sujets : l'echantillon examiné comprend 461 élèves issus des quatre niveaux scolaires du premier cycle de l'enseignement secondaire français (niveau sixième : âge normal 11-12 ans; cinquième : 12-13 ans; quatrième 13-14 ans; troisième : 14-15 ans).

Matériel : Le matériel graphique est composé de diagrammes en bâtons. Son caractère élémentaire -une seule variable à une dimension- n'induit toutefois pas des traitements eux-mêmes élémentaires (cf. Inhelder, 1970; Baillé, Maury, Janvier, 1988).

Quatre séries de quinze items chacune sont organisées. Tout item (cf. exemples ci-après) comprend des nombres qui représentent des populations de villes et un diagramme. Relativement aux valeurs numériques et aux longueurs des bâtons, toutes les séries sont constituées sur le même modèle: cinq items présentent des rapports "scalaires" simples, entre des grandeurs de même nature, nombres ou bâtons ; cinq autres des relations de type "fonction" simples entre des grandeurs de natures différentes, enfin les cinq derniers contiennent des relations additives simples entre grandeurs de même nature.

Les séries se distribuent en deux classes qui renvoient aux activités présupposées de lecture (L) et de construction (C). L'activité (L) est dirigée soit vers les nombres (N) soit vers les diagrammes (D). Il en va de même pour (C). On a donc les quatre séries suivantes :(LD); (LN); (CD) et (CN). A titre d'exemple, nous avons reproduit en page suivante le premier item de chaque série (figure 1).

item N°1
de la
série LD

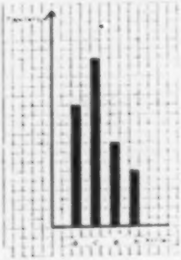


Voici 4 villes dont les populations sont :
A : 6 000 habitants; B : 18 000 habitants
C : 9 000 habitants; D : 13 000 habitants

Sur le graphique on a déja indiqué le bâton qui représente la population de la ville B.

Placez sous les autres bâtons le nom des deux autres villes qui ont été représentées.

item N°1
de la
série LN



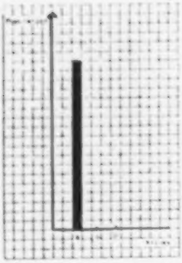
Voici un graphique qui représente les populations de 4 villes A,B,C,D.

On sait déjà que le bâton C représente une ville de 18 000 habitants.

Indiquer à quels bâtons du graphique correspondent les populations suivantes :

9 000 habitants :
6 000 habitants :

item N°1
de la
série CD

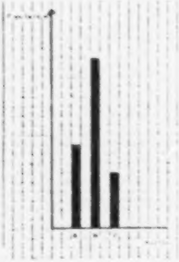


Voici 3 villes dont les populations sont :
A : 18 000 habitants; B : 6 000 habitants
C : 9 000 habitants

Sur le graphique on a déja indiqué le bâton qui représente la population de la ville A.

Tracez au dessus des noms des deux autres villes les bâtons qui les représentent.

item N°1
de la
série CN



Voici un graphique qui représente les populations de 3 villes A,B,C.

On sait déja que le bâton B représente une ville de 18 000 habitants.

Indiquez les populations des villes représentées par les bâtons suivantes :

A :
B :

Fig.2 : item N°1 de chacune des séries

Le plan d'expérience combine trois facteurs : le niveau scolaire (quatre modalités : sixième, cinquième, quatrième, troisième), un facteur activité (A) à deux modalités (L) et (C), un facteur "objet" vers lequel est dirigé l'activité (D) et (N). La variable dépendante étudiée est la performance : chaque item est noté 0 en cas d'échec, 1 pour une réussite partielle et 2 pour la réussite totale.

Dans chaque niveau scolaire, les élèves sont répartis au hasard en quatre groupes. Chaque groupe correspond à une disposition (LN), (LD), (CN), (CD) des modalités. La durée de l'épreuve est limitée à 50 minutes.

Résultats :

a) *Etude globale*

Pour chaque niveau scolaire, nous avons porté dans le tableau ci-dessous les moyennes obtenues par chacun des groupes (moyennes sur 30 points, arrondies au point près).

Tableau I

Moyenne par niveau scolaire et par groupe

Niveau scolaire	sixième				cinquième				quatrième				troisième			
Groupe	CN	CD	LN	LD	CN	CD	LN	LD	CN	CD	LN	LD	CN	CD	LN	LD
Moyenne	15	16	22	19	15	18	25	22	22	21	27	25	22	23	28	26

Afin de tester l'effet des trois facteurs ainsi que les éventuelles interactions, nous avons effectué une analyse de la variance sur les données résumées dans le tableau I. A cet effet, nous avons utilisé la procédure G.L.M. (General Linear Models) du système Statistical Analysis System (S.A.S. 1982), en demandant en option le test des rangs multiples de Duncan (ce test permet de regrouper les moyennes en classe à l'intérieur desquelles les différence ne sont pas statistiquement significatives). Des résultats de l'analyse, portés dans le tableau II, attestent le fort impact du niveau scolaire sur les performances des élèves. Toutefois, nous verrons que la progression observée ne renvoit pas strictement à la suite des niveaux scolaires. Notamment, l'application du test de Duncan conduit à une indifférenciation des niveaux troisième et quatrième. Se distinguent et suivent dans l'ordre d'une performance décroissante, les élèves de cinquième et enfin ceux de sixième.

Tableau II

Résultats de l'analyse de la variance relative aux données du tableau I

Source	DF	F	PR>F
Niveau scolaire	3	25.58	0.0001
Activité (L, C)	1	102.12	0.0001
Objet (N, D)	1	0.34	0.3611
Activité x objet	1	9.78	0.0019
Classe x activité	3	1.35	0.2562
Classe x objet	3	0.12	0.2470
Classe x objet x activité	3	1.09	0.3516

Ajoutons que si les activités (L) et (C) se distinguent, l'activité de lecture étant très significativement plus facile, tous objets confondus que celle de construction, il semble, en revanche, que la nature (D) ou (N) de l'objet visé ne détermine aucune différence significative des performances. Il reste que la forte interaction observée entre objets et activités méritera, plus loin, quelques commentaires.

b) Etude par modalité du facteur activité

Tableau II

Moyennes des notes (M) et écarts types (σ) par niveau scolaire pour chacune des activités (L) et (C), tous objets confondus

Niveaux scolaires	sixième		cinquième		quatrième		troisième	
Activité	L	C	L	C	L	C	L	C
M	20,7	15,6	23,5	16,4	25,9	21,1	26,7	22,2
σ	6,6	5,8	5,5	6,9	4,9	6,8	3,2	6,7

A chaque niveau, tous objets confondus, il est manifestement plus facile de lire que de construire. Du point de vue des groupements de Duncan, l'activité construction et l'activité lecture déterminent, chacune, un groupe de deux classes indifférenciées : la quatrième et la troisième. Mais, alors que l'activité lecture conduit à des performances moyennes distinctes entre la sixième et la cinquième, l'activité construction ne partitionne pas ces deux premiers niveaux scolaires.

Par rapport à l'efficacité du traitement de ces diagrammes en bâtons, il semble donc que le niveau quatrième corresponde à un saut significatif. D'autre part, les résultats suggèrent qu'une attention plus grande soit portée à l'apprentissage de la construction de représentations graphiques dans les premiers niveaux de l'enseignement secondaire.

Afin de faciliter l'interprétation de l'interaction mise en évidence par l'analyse de la variance (tableau II), nous avons construit la figure 2.

c) Etude de l'interaction entre objets et activités

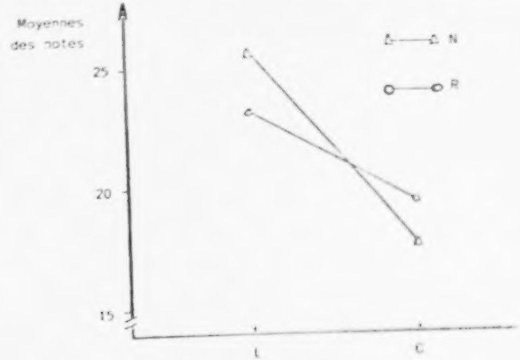


Fig.2 : représentation de l'interaction

Relativement au gain de lisibilité de la représentation graphique (cf. Bertin, 1979), ce résultat paraît surprenant, au premier abord. Mais, d'une part, la suite des nombres est suffisamment courte pour se situer dans les limites de l'empan en mémoire de travail et, d'autre part, l'organisation du matériel n'est pas sans incidence. Dans les items de la série (LN) où la valeur des nombres tend à faciliter le recours aux procédures scalaires ou additives (cf. Vergnaud et al, 1979), la disposition des nombres renforce visuellement ce type de traitement. Dans les items correspondants de la série (LD), les élèves doivent, au préalable, mesurer la longueur des bâtons. Cette interprétation présuppose un ancrage initial du traitement sur les nombres en (LN) et sur le diagramme en (LD).

Dans l'activité de construction, le risque d'erreur n'est pas équivalent sur les bâtons et les nombres. Sur les bâtons, l'erreur est plus rare car plus nette (au minimum un carreau en plus ou en moins). L'ajustement au carreau près peut conduire à une réponse juste des élèves ayant adopté une procédure retour à l'unité même quand l'échelle ne conduit pas à un nombre entier de carreaux par millier d'habitants. Dans le cas où les élèves adoptent la même procédure dans les items correspondants de la série (CN), ils calculent la "valeur" (en nombre d'habitants) de 1 carreau. Celle-ci est un nombre décimal non entier qu'ils multiplient ensuite par le nombre de carreaux. Ils obtiennent un nombre dont ils ne retiennent que la partie entière. Cet ajustement conduit dans tous les cas à une erreur alors qu'il n'en va pas de même lors de l'ajustement au carreau près.

Discussion

Dans un commentaire qui succédait à la présentation des résultats concernant l'effet du facteur activité, nous avons déclaré que les activités de lecture étaient manifestement plus facile que les activités de construction. Il est vrai que les résultats se prêtent à ce rapide commentaire. Mais celui-ci n'a de sens que si les deux opérations, en dehors du fait qu'elles impliquent les mêmes objets, sont comparables. Sans doute pour les nombres comme pour les diagrammes, en lecture et en construction, le traitement implique-t-il le calcul, direct sur les premiers et par le biais de la mesure sur les seconds. Cependant, en lecture, nous pouvons supposer que la composante perceptivo-cognitive de la mémoire de travail facilite une estimation comparative de la vraisemblance des calculs. En revanche, la construction n'autoriserait pas cette recherche implicite de vraisemblance au long des calculs. Dans ce dernier cas, si vérification il y a, elle n'opérerait qu'au terme des calculs, sans possibilité de comparaison. Ajoutons que l'interaction témoigne d'une complexité plus grande du phénomène.

Pour conclure, soulignons que cette simple exploration des incidences séparées et conjointes des objets et des activités montre l'intérêt d'une diversification des exercices préparés aux élèves, si l'on veut que l'interprétation des graphiques se fonde sur une réversibilité complète du couple données-diagrammes.

Références

Baillé J., Maury S., Janvier M. : 1988, les représentations graphiques dans le premier cycle de l'enseignement secondaire. Rapport de recherche. Ministère de l'Éducation Nationale.

Bertin J. : 1977, la graphique et le traitement graphique de l'information. Paris : Flammarion.

Inhelder B. : 1970, "Mémoire et intelligence" in Association de Psychologie Scientifique de langue française, ed. *La Mémoire*, Paris, PUF.

S.A.S. : 1982, *User's guide* : *statistics*. S.A.S. Institute Inc : Cary, North California.

Vergnaud G. et al. : 1979, acquisition des "structures multiplicatives" dans le premier cycle du second degré. Orléans : IREM, Paris E.H.E.S.S.

COOPERATIVE GROUP LEARNING OF GEOMETRIC PROOF CONSTRUCTION:

A CLASSROOM ASSESSMENT

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In this study, the effects of two different teaching methods on achievement in and attitude towards the learning of deductive geometry were examined. The experimental method emphasized small-group, cooperative learning through process-oriented, proof-construction tasks designed in accordance with the van Hiele developmental levels. The control method, direct instruction, emphasized whole-class, teacher-led instruction. No significant differences in overall achievement or attitude were found between the two treatment groups. However, the cooperative-learning group did make modest process-oriented gains while maintaining conventional Grade 10 geometry skills. The results of a descriptive item analysis suggest that while the direct instruction method tended to produce a higher rate of student success on items that tested straight knowledge and application of the geometric content studied, the cooperative learning method tended to enhance student performance on actual proof construction.

The current research literature on the teaching of geometry indicates that high school students are mostly unsuccessful in their efforts to learn proof construction (Senk, 1983, 1985) and that high school geometry as experienced by many students, if not most, is but a collection of meaningless isolated facts and proof sequences to be memorized in order to pass the exams (Hoffer, 1981; McDonald, 1983; and others). Is it possible to teach geometry and proof construction to high school students so that higher levels of thinking in mathematics are cultivated rather than rote memorization of facts? To assess the feasibility of one learning-theory-based alternative to conventional approaches to teaching deductive geometry, MacRae (1988) compared the effects of cooperative group learning of process-oriented tasks with those of direct classroom instruction.

Questions Investigated

1. Will there be a significant difference between the group mean post-test and retention-test achievement scores obtained by the students taught geometry with direct instruction and those taught geometry in cooperative-learning groups using process-oriented materials?

2. Is one method superior to the other for producing a higher student success rate on particular types of geometry or proof questions?

3. To what extent will either method be successful in teaching students to solve standard geometrical problems and to construct proofs at various difficulty levels?

4. What kinds of attitudes will be exhibited towards geometry as measured by enjoyment and anxiety subscales, before and after the study of the geometry unit? Will there be significant differences between the group mean changes in attitude?

Method

Procedure

Two 25-student Mathematics 10 classes participated in the study. One class studied geometry in a guided discovery approach, using cooperative learning techniques (e.g., Slavin, 1980; Sharan & Sharan, 1976) and process-oriented geometry materials which were structured around the van Hiele levels of mental development (Freudenthal, 1973). A direct-instruction method was used in teaching the other class, emphasizing those teaching behaviors currently held to be most effective for student learning under direct instruction, namely: emphasis on active teaching with little seatwork, frequent feedback through regular homework checking, smooth transitions between activities, clear presentations with explanations of each step in the learning process, regular review, and a fairly fast paced delivery (Good & Grouws, 1977; Good, 1982; Brophy & Good, 1984; Good, 1984).

The study began after a common midterm algebra exam was given. No significant difference between the class means of the two groups on the midterm exam was found.

Both classes were taught the Mathematics 10 geometry unit by the first author over a period of four weeks. The members of the "direct-instruction" class had ample time to finish and review the geometry course in preparation for the achievement exams written at the end of the study. The cooperative-learning group was hard pressed to cover the content in the four weeks of the study so no review or practice time was allowed them before the exams were written.

Upon completion of the unit both classes wrote a geometry unit exam and a proof test. Seven months later a geometry retention exam was given to 14 students from each of the original treatment groups who could be located in a sequent mathematics course. Both classes also wrote pre- and post- attitude towards geometry questionnaires to indicate change in enjoyment and anxiety towards the learning of geometry.

Analysis

The data collected from the achievement tests were analyzed using t-tests to determine whether there were significant differences between the class means of the two groups on each test. The results of the attitude questionnaire were analyzed using a two-factor analysis of variance with repeated-measures design.

A descriptive analysis of specific items on the achievement tests was used to determine whether the students who were taught through a cooperative-learning method which used process-oriented materials attained a higher percentage of successes or failures on particular types of geometry questions than the students taught by direct-instruction. An analysis of student responses to these items revealed patterns indicating specific areas of strength and weakness of teaching method with respect to the teaching of proof construction.

Results

The results of the statistical analyses indicated that there were no significant differences between the group means of the direct instruction class and the cooperative-learning class on the three geometry achievement tests given. This is an interesting result considering that the cooperative-learning class was severely pressed to finish the unit in time to write the exams and, unlike the direct-instruction class, was given no time in class to review or prepare for the exams.

However, there is reason to believe that the cooperative-learning, "guided discovery" group did make some modest process-oriented gains while maintaining conventional Grade 10 geometry skills. The results of a descriptive item analysis suggested that while the direct instruction method seemed to produce a higher rate of student success on items that tested straight knowledge and application of the geometric content studied, the cooperative-learning method enhanced student performance on actual proof construction.

The results of the statistical analysis of the Attitude Towards Geometry questionnaire indicated that both groups showed a slight improvement in enjoyment of geometry and a slight lowering of anxiety towards geometry at the conclusion of the study but there were no significant differences between the group mean scores.

Discussion

Strengths and Weaknesses of Each Method

The strengths of the cooperative learning method included: providing for a closer contact with the geometry concepts and a greater involvement in forming one's own procedures for constructing proofs, promoting intellectual involvement through the exchange of ideas with others to help clarify one's thinking and thus understanding, helping students appreciate what is involved in learning axiomatic systems, and encouraging students to be more independent of the teacher in learning mathematics. In particular, the members of the cooperative-learning class seemed to be more adventuresome than those in the direct-instruction class in attempting proofs to the more difficult and unfamiliar problems, even though they may not have been particularly successful in completing them. They were quite used to attacking problems and creating their own procedures for solving them without teacher guidance.

Another cooperative-learning strength was that the students using this method seemed to enjoy working and learning in small groups. However, it is surprising that their measured change in attitude towards geometry did not seem to reflect this enjoyment. A reason for this finding might be that while the students did enjoy the group work, they did not necessarily enjoy learning geometry as much. They found geometry and especially proof construction much more difficult than algebra. Furthermore, not having had the time to complete the unit by using the cooperative-learning method or to consolidate their knowledge through a review of the material in preparation for the exams could well have left them less happy with their accomplishments than they may otherwise have been.

The greatest weakness of the cooperative-learning method seems to be the amount of time required to implement it in a classroom situation. Not enough time was devoted to review and practice of the material learned. There are other weaknesses which might have been overcome with more teacher guidance. First, not enough attention was given to making the "discovered" concepts explicit (especially to the less able members of the class) and to the details of setting up the form in which a proof should be presented. Second, the lack of taught procedures which could be applied to new problems so that the students did not have to start from first principles on each question may have slowed them down when writing exams. However, if more time had been available for review and practice this weakness may have been overcome. Third, it was perhaps too easy for the less able students in each group to hide behind the accomplishments of the more able group members to the detriment of their own mathematical development. Fourth, it is possible that at least some of the "discoveries" were not actually made by the students in their group discussions. In this case they would not have learned the relevant geometric concept or relation well enough to apply it to other problems.

The strengths of the direct-instruction method seem to follow directly from the weaknesses of the cooperative-learning method: more teacher guidance in concept formation and proof construction, more time available for review and practice of the material studied, and more teacher help available especially for the less able students. Additionally, the students in the direct-instruction class might have been be better prepared for the Math 10 Geometry Exam, in particular, since in this approach teachers tend to (inadvertently) teach towards the exam; that is, to cover explicitly all of the content most likely to be tested.

A major weakness of the direct-instruction method is that students become too dependent on the teacher in their learning of mathematics. The members of the direct-instruction class were much more insecure than those of the cooperative-learning class when faced with a problem, especially a proof, they had not seen before. They relied too heavily on the teacher to show them procedures which they could use, before they had even attempted the question. The cooperative-learning members, on the other hand, would complete the proof to the question first, and then ask the teacher to check the finished product.

Another weakness of the direct-instruction method is that the teacher can never really tell whether the students have understood the concepts presented, or whether they have just memorized them. And, in using the question-answer technique (Socratic approach), it is the teacher who is giving the leading questions based on a logical approach to the problem -- the students supply answers but perhaps never really learn which pertinent questions they should be asking themselves when attacking problems on their own. The ability to make correct replies to leading questions is not necessarily indicative of the students having a clear understanding of what is involved in proof construction. One of the major problems with the direct instruction method in teaching geometry is that the teacher can never be certain about what and how the students are learning. What are the students' perceptions of the material that is delivered to them? Do they really understand what is being taught well enough to be able to apply it, or do they resort only to memorizing what they are taught? Are they becoming too dependent on taught procedures and methods to the detriment of learning their own? While it is true that students learning through the cooperative learning method may also resort only to memorizing facts and approaches, it is perhaps less likely because they are encouraged to actively negotiate their mathematical knowledge through discussion with their group members as well as the teacher. Perhaps a synthesis of the two methods which draws on the strengths of each could be developed and tested in another study.

Discussion of the Achievement Tests

The Math 10 Geometry Exam (observed reliability coefficient of 0.7919) was a standard Mathematics 10 geometry exam which adequately tested the objectives as listed in the provincial curriculum. The item analysis revealed different strengths and weaknesses in addressing particular items which may have been the direct result of the teaching methods employed. But the Geometry Proof Test (observed reliability coefficient of 0.7302) was perhaps far too difficult to expose many of the real differences between the two groups in actual understanding of proof construction. It did reveal the students' inadequacies and misconceptions in the different areas of geometry more clearly than did the former exam. The results suggested that the problems that students had with proof construction were not necessarily of a logical nature but had more to do with lack of intuition of the geometric concepts and relations themselves. To rectify this more time would be required for students to work with the concepts and to apply them in various situations.

The Geometry Retention Test (observed reliability coefficient of 0.9999) was taken by the students seven months later and without any preparation for it. The knowledge and application questions were suitable for testing the amount of geometrical knowledge retained by the members of both groups, but the proofs may have been too difficult to indicate adequately retention of proof construction. Perhaps some straight-forward proofs which involved simpler, more basic concepts should have been included, to better indicate how far the students could still go towards setting up a proper proof.

Conclusion

The cooperative-learning approach using process-oriented geometry materials structured around the van Hiele levels of mental development can add an important dimension to the learning of geometry which is not necessarily experienced by students taught exclusively by the direct-instruction approach; that is, an active participation of the learners in the exploration of geometrical concepts, leading to the construction of their own axiomatic systems of geometry. It is a method which, with appropriate modifications designed to address the weaknesses found in this study, can be used to provide personal experiences of "mathematizing," and thus to cultivate higher levels of thinking in mathematics. It can be argued that a cooperative-learning approach directly addresses concerns about student beliefs that "formal mathematics and proof have nothing to do with discovery and invention," resulting in failure to use the results of formal mathematics in problem-solving situations (Schoenfeld, 1985, 1987). And it is interesting to note that once the students had experienced the cooperative-learning method many were reluctant to go back to the direct instruction method at the conclusion of the study. They preferred to discover for themselves and at their own rate, and not to be "told" or to have to work things out in whole class discussions where many of them could not get the chance to explore their own ideas.

Because of the limited time allotted to the study of geometry in a Mathematics 10 course, it is, perhaps, not an especially practical classroom approach. Instead it may be wiser to devise a teaching approach which attempts to draw on the strengths and eliminate the weaknesses of both methods, for teaching high school geometry. The process-oriented materials and group learning sessions could be used in conjunction with more frequent teacher-led class sessions to consolidate the understanding of the concepts and procedures "discovered."

It may be that too much was expected from the students in a relatively short geometry unit, especially as far as the level of thinking and proof construction demanded on the achievement tests were concerned. While the exams may have been better constructed, the researcher believes that the intellectual demands made on the students and depth of the material covered and tested should not be lessened in an attempt to make for better exam results. The students need real experiences of "mathematizing," and if more time is required for this to be realized then perhaps one should take a serious look at mathematics curricula to see how this might be accomplished.

References

Brophy, J. & Good, T. (1984). Teacher behavior and student achievement. (Occasional paper #73). The Institute for Research on Teaching, Michigan State University, East Lansing. (ERIC document ED 251 422)

Freudenthal, H. (1973). Mathematics as an Educational Task. Dordrecht, Holland: D. Reidel.

Good, T. (1982). Classroom research: what we know and what we need to know. Austin, Texas: University of Texas, Research and Development Centre for Teacher Education.

Good, T. (1984). Teacher effects: Making our schools more effective. Proceedings of Three State Conferences. Columbia: University of Missouri. (ERIC document ED 249 585).

Good, T. & Grouws, D. (1977). Teaching effects: a process-product study in fourth grade mathematics classrooms. Journal of Teacher Education, 28 (3), 49-54.

Hoffer, D. (1981, January). Geometry is more than proof. The Mathematics Teacher, 11-18.

MacRae, L. (1988). Assessing the Relative Effectiveness of a Cooperative Learning Approach to Teaching Grade Ten Geometry. Unpublished master's thesis, University of Calgary, Calgary, Canada.

McDonald, J. (1983, June). Formal operations and the ability of students to structure geometric content. Paper presented at the Annual Meeting of the Jean Piaget Society.

Schoenfeld, A. (1985, March). Student beliefs about mathematics and their effects on mathematical performance: a questionnaire analysis. Paper presented at the Annual Meeting of the American Educational Research Association.

Schoenfeld, A. (1987). Confessions of an accidental theorist. For the Learning of Mathematics, 7 (1), 30-38.

Senk, S. (1983). Proof-writing Achievement and van Hiele Levels among Secondary School Students. Unpublished doctoral dissertation, University of Chicago.

Sharan, S. & Sharan, Y. (1976). Small Group Teaching. New Jersey: Educational Technology Publications.

Slavin, R. (1980). Cooperative learning. Review of Educational Research, 50, 315-342.

COMPARING EXPERTS' AND NOVICES' AFFECTIVE REACTIONS TO MATHEMATICAL PROBLEM SOLVING: AN EXPLORATORY STUDY

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Abstract

This study investigated the role of affective factors in the performance of both experts and novices who were asked to solve nonroutine mathematical problems. The affective reactions of the experts (four research mathematicians) were similar to those of the novices (four undergraduate students), but experts and novices differed in their ability to control the influence of affective factors.

INTRODUCTION

Research on mathematical problem solving has tended to concentrate on cognitive factors that influence performance. In recent years, however, there has been increasing recognition of the importance of affective factors in problem solving (Silver, 1985). In this study we investigated the differences between experts and novices in their affective responses to nonroutine mathematical problems.

THEORETICAL BACKGROUND

Research on the affective domain has generally emphasized the use of questionnaire data related to beliefs and attitudes. In mathematical problem solving, however, affective responses can be much more intense and emotional, rather than attitudinal, in nature (Mason, Burton, & Stacey, 1982; McLeod, 1988). The investigation of affective reactions to mathematical problem solving requires a new theoretical foundation if it is to proceed in an intellectually satisfying way. The work of Mandler (1984, in press) provides such a foundation.

Mandler's view is that the basis of affective reactions to problem solving (or more generally any task) is generated out of the solver's emotional responses to the interruption of a plan. In Mandler's terms, plans arise from the activation of a schema. The schema produces an action sequence; if the anticipated sequence of actions cannot be completed, the blockage or discrepancy is followed by the arousal of the autonomic nervous system. This response may be experienced as an increase in heartbeat or in muscle tension. The arousal serves as the mechanism for alerting the individual and redirecting attention to the source of the interruption. When the arousal occurs, the individual attempts to evaluate the meaning of the interruption. This interpretation of the interruption might classify it as a frustrating block or perhaps a challenging surprise. The cognitive evaluation of the interruption provides the meaning to the arousal.

In mathematics education, problems are usually defined as those tasks where some sort of blockage or interruption occurs. The student either does not have a routine way of solving the problem, or the routine attempts to solve the problem all fail. As a result, the kind of problem solving that is attempted by mathematics students results in just the kind of interruption that Mandler has analyzed in his theory. In this study the responses of both experts and novices were examined to look for similarities and differences in their responses to interruptions in problem solving.

DESIGN AND PROCEDURES

Interviews were conducted with eight subjects who had participated in problem-solving sessions. Four of the subjects were research-active professors of mathematics; all were males in the middle of their careers. The other four subjects were undergraduate students (two female, two male) who were majoring in one of the social sciences or education; all were enrolled in a college-level mathematics course at the time. All interviews lasted about an hour; most subjects participated in two interviews. The four professors were considered "expert" problem solvers, and the four students were designated "novice" problem solvers.

The four professors were originally chosen to participate in a study of how aesthetic factors influenced problem-solving performance among eight mathematicians and graduate students (Silver & Metzger, in press). These four were chosen for this study since they were the most senior and experienced problem solvers in the sample. The four undergraduates were chosen from a set of six volunteers based on whether or not they were successful on at least one of the problems.

The interviews began with a discussion of the think-aloud procedure for gathering data on thought processes during problem solving (Ericsson & Simon, 1980). Then participants were asked to think aloud as they solved problems. After they had attempted one or more of the problems, the subjects were asked to comment on their feelings about problem solving, particularly their feelings about being stuck on a problem or how they felt when they had solved a problem. After the discussion of affective factors related to problem solving, the subjects went on to solve additional problems. In some cases the discussion returned to the topic of affect and problem solving.

Different problems were chosen for the experts and the novices. A sample problem from the expert category is the following (Schoenfeld, 1985):

> Three points are chosen on the circumference of a circle and the triangle containing them is drawn. What choice of points results in the triangle with the largest possible area? Justify your answer.

Here is a typical problem from the novice category:

> A man entered an orchard through seven gates and there took a certain number of apples. As he left the orchard, he gave the first guard half the apples that he had and one apple more. To the second guard he gave half his remaining apples and one more. He did the same to each of the remaining five guards and left the orchard with one apple. How many apples did he gather in the orchard?

For a complete list of problems given to the experts, see Silver and Metzger (in press). The problems given to the novices were similar to those in Burton (1984).

The novices solved 6 out of 13 problems; the experts also solved about half of their problems (8 out of 17). The problem set appeared to achieve the right level of difficulty for each group; all subjects experienced some success and some failure as they tried to solve the problems.

RESULTS

Audiotapes of the interviews were transcribed and checked for accuracy by the experimenters. These protocols constituted the data for the study. The analysis of the data focussed on the magnitude, direction, awareness, and control of the emotional reactions that were reported by the subjects (McLeod, 1988).

Most experts and novices reported that they experienced relatively intense emotional reactions when solving problems. When they were asked to describe their feelings when stuck on a problem, experts used words like frustration, aggravation, and disappointment. Novices expressed many of the same feelings, but they also referred to themselves using words like dumb and stupid. The novices' negative statements about their own ability suggest that their causal attributions may be quite different from those of experts (Fennema & Peterson, 1985; Heckhausen, 1987). Although novices tended to make more negative comments than experts, both groups reported a mix of rather strong positive and negative emotional reactions to problems. One expert noted that as a research mathematician, he was stuck on one problem or another all the time; perhaps as a consequence of this view, his comments on his emotional reactions to problem solving were somewhat less intense than the other experts. One novice also seemed to have less intense reactions; he reported that he got frustrated, but didn't feel too badly since mathematics was not his emphasis.

Both experts and novices indicated considerable awareness of their emotions during problem solving. One of the experts commented on his nervousness, and compared it to how he felt when doing unfamiliar problems in front of the class. A novice noted how he got worried when the numbers in the seven gates problem started getting too large. When novices were asked about their frustrations, they generally indicated that they were aware of when they were frustrated. Both experts and novices indicated that they had certain preferred strategies when they became aware of their frustrations; the most popular strategy was to quit working and come back to the problem later. However, experts were more likely to suggest other strategies like using special cases or visualization.

Experts and novices showed substantial differences in their ability to stay in control of their emotions during problem solving. Experts were more likely to comment on the need to stay flexible, especially when stuck and frustrated with a problem. The novices were more likely to get stuck "in a groove" and keep on trying to solve the problem in the same way. For example, novices would repeatedly try to represent the seven gates problem with an equation rather than changing to a more helpful strategy like working backwards. We hypothesize that novices were more likely to use up short-term memory in evaluating their affective responses, thus reducing their ability to think of new approaches to the problem.

In summary, the experts and novices in this study responded to problem solving tasks with strong emotions, both positive and negative. Novices more frequently expressed negative feelings about their own performance. Both experts and novices were aware of their own emotions, but the experts were more likely to remain in control. Novices appeared to be more likely to let their frustrations drive them to repeated use of the same strategy, but experts were more likely to stay flexible and to consider alternate strategies.

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REFERENCES

Burton, L. (1984). *Thinking things through: Problem solving in mathematics*. Oxford: Basil Blackwell.

Ericsson, K. A., & Simon, H. A. (1980). Verbal reports as data. *Psychological Review*, 87, 215-251.

Fennema, E., & Peterson, P. (1985). Autonomous learning behavior: A possible explanation of gender-related differences in mathematics. In L. C. Wilkinson & C. Marrett (Eds.), *Gender influences in classroom interaction* (pp. 17-35). Orlando: Academic Press.

Heckhausen, H. (1987). Causal attribution patterns for achievement outcomes: Individual differences, possible types, and their origins. In F. E. Weinert & R. H. Kluwe (Eds.), *Metacognition, motivation, and understanding*. Hillsdale, NJ: Erlbaum.

Mandler, G. (1984). *Mind and body: Psychology of emotion and stress*. New York: Norton.

Mandler, G. (in press). Affect and learning: Causes and consequences of emotional interactions. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective*. New York: Springer-Verlag.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. London: Addison-Wesley.

McLeod, D. B. (1988). Affective issues in mathematical problem solving: Some theoretical considerations. *Journal for Research in Mathematics Education*, *19*, 134-141.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando: Academic Press.

Silver, E. A. (1985). Research on teaching mathematical problem solving: Some underrepresented themes and needed directions. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 247-266). Hillsdale, NJ: Erlbaum.

Silver, E. A. & Metzger, W. R. (in press). Aesthetic influences on expert mathematical problem solving. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective*. New York: Springer-Verlag.

THE DEVELOPMENT OF CHILDREN'S CONCEPTS OF ANGLE

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The literature on the development of children's perception of ray pairs, regions and rotations is summarised and related to research on the geometrical concept of angle. Implications are drawn for teaching and research.

Classroom researchers repeatedly report how difficult children find the angle concept. One reason is certainly its many-sided nature, which in mathematics must be unified into a single definition. Practical experiences, as well as mathematical definitions, tend to fall into three categotries:

1. a ray-pair giving the difference between two directions;
2. a region of a plane bounded by two rays with a common end-point;
3. an amount of rotation.

The purpose of this paper is to describe the development of children's understanding of these three concepts; to draw implications for geometry teaching; and to indicate research needs. To save space, only a few key studies will be cited by name; a full bibliography is available from the author.

PERCEPTUAL RESEARCH

We look first at research in which the word "angle" is not used in its mathematical sense, the task being to process a given figure which happens to contain angles. Basic to much of this research is the concept of orientation. It has been demonstrated that equality of orientation plays a fundamental role in perceptual processing. However, orientation has a low saliency in deciding when two figures are "the same", so that responses are sensitive to a strong bias towards orientations in which a figure appears upright or stable or aligned with a surrounding frame. Vertical and horizontal orientations thus play a privileged role in perception.

Angle as ray pair Very young children can discriminate the orientation of single lines (which we may regard as the angle with the perceptual vertical), but it is only at age 7 that children demonstrate the ability to memorise orientations. Bryant (1974) claimed that 5-yr-olds could easily discriminate right angles from non-right angles and do this better than they discriminate non-right angles from each other. However, Noss (1987) found that less than 50% of his 10-11-yr-olds thought that the two angles in Fig 1 were equal.

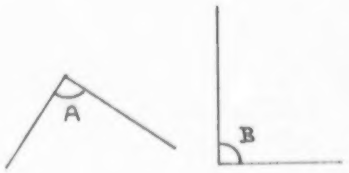

Fig 1

Copying of single lines and angles is subject to systematic biases which depend on line orientation, angle size and the orientation of the surrounding frame, and which decrease with age.

In most studies, it is not clear whether children are processing the depicted angle or the entire figure. An exception is a study of Piaget et al. (1960), in which children copied Fig 2. Up to age 8, children used mainly visual estimation; older children usually measured AD, DB, DC,

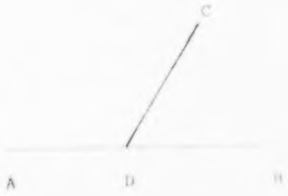

Fig 2

AC and CB, apparently copying the entire figure. However, children aged 10 and over often measured AD, DB, DC and the perpendicular from C to AB, suggesting that they were attempting to copy the angle CDB.

Several studies suggest that angles are not salient properties of figures. For example, young children often prefer to preserve the topological properties of a figure rather than copying its angles, and they do not copy parallel lines parallel in the presence of a distracting background of oblique lines.

Angle as region The idea of an angle as a region has mostly only been studied incidentally, in that the sharpness of a corner can be a distinguishing feature of a shape. It is generally found that the accuracy and speed of recognition varies with the angular displacement between stimulus and response and that performance improves with age and training.

Beilin (1979) investigated children's concepts of angles as regions. He found that even 4-yr-olds could mark corresponding angles of congruent triangles without difficulty. However, in checking whether two triangles were congruent, young children only tested the sides and never the angles; the use of superposition only became popular from age 7. It is nevertheless surprising that even 12-yr-olds have difficulty deciding whether a square template fits into a given angle (Wallrabenstein, 1973).

Angle as rotation Rotation is in itself a difficult transformation: non-conservers of length believe that rotation changes lengths, and it is not until age 10-11 that more than 50% of children can represent the rotation of a simple figure about a vertex. It may be noted that mental rotation is at the basis of many spatial ability tests, and shows wide individual variation. Speed of processing seems to be linearly dependent on the angle of rotation.

RESEARCH ON ANGLES

We turn now to research where the word "angle" has to be interpreted in its mathematical sense (as ray pair, region or rotation) and not simply as a figure. We can summarise the sparse research here according to a few common findings.

Right angles are often not recognised in oblique position. Children at first only accept the case of an angle between a horizontal and a vertical line as a right angle. Even at age 11, only about 50-60% of children accept both figures in Fig 1 above as right angles.

Angle size is often believed to depend on the lengths of the arms. This error seems to be the result of treating the given figure as finite and complete. Noss (1987) found that the experience of rotations in LOGO helped improved 10-yr-olds' performance on the angle comparison task in Fig 3 - but not on that in Fig 1. However, it is also possible that asking "which angle is sharper?" instead of "which angle is bigger?" would also have lead to improved performance.

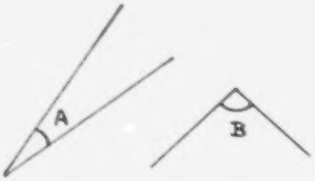

Fig 3

Reflex angles are recognized late. Early angle concepts seem to be restricted to convex angles (those up to 180 degrees in size). For example, Close (1982) found several children who believed that the angle in Fig 4 was obtuse. This could be the result of limited experience of concave corners, or because the figure is not seen as a rotation.

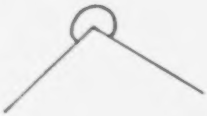

Fig 4

Measuring angles is difficult to learn. Every teacher knows this! But measuring angles requires the ability to combine unit angles and compare the result with a given angle - apparently using regions, but often explained using rotations and later applied to ray pairs. In order to understand angle measurement, the child must not only have mastered all three aspects of the angle concept but also have achieved a high degree of integration between them.

SUMMARY: DEVELOPMENT OF THE ANGLE CONCEPT

Children's perception of angles develops rapidly from the age of 5 years. Children must learn to overcome orientation biases, develop memory for orientation, isolate corners as distinguishing features of shapes, and learn to represent rotations. Not until 12 years of age can one expect the majority of children to have completed this process to a satisfactory degree.

A critical step in the subsequent abstraction of the mathematical concept of angle would seem to be the establishment of an angle as a class of equivalent figures. I conjecture that this takes place independently for ray pairs, regions and rotations. Consider, for example, ray pairs. At first only angles related by a translation are accepted as "the same"; then angles related by a reflection in a vertical or horizontal line; and only much later those related by a general isometry combined with an arbitrary extension or contraction of the arms. Notice that it is not sufficient for the child to recognize that two angles are equal, although this skill must certainly be learnt; it is the acceptance of equal angles in all orientations as "the same" which shows that the concept of angle has been abstracted.

An important development is the integration of the three angle concepts. How this occurs is completely unknown, but one can speculate that the ray pair (as an abstract representation of both regions and rotations) plays a central role. The integration of the region and rotation aspects of angle probably takes place via the ray-pair aspect and would be a rather late development.

IMPLICATIONS FOR TEACHING

We have reached the position that the geometrical concept of angle cannot be treated as either ray-pair or region or rotation, but must be an integration of all three. Teaching should therefore encourage the development of all three aspects of angle, providing ever closer connections between them until the student is ready to treat the concept as a unity with many applications.

The concept of angle as region tends to be treated in the elementary school and to be restricted to the right angle. Further activities comparing the "sharpness" of corners of figures by superposition and on fitting corners together (as in tessellations) could be highly beneficial. Figures with concave corners should not be omitted. Diagrams used in this work should show the angle as a shaded region with the suggestion of unboundedness.

The angle as rotation on the other hand tends to be left for the secondary school, just before angle measurement. This is certainly too late: informal experience of rotation can and should be gained in the elementary school (Kirsche, 1987). Rotation angles may be conveniently represented by clocklike diagrams, but the clock should not always start at 12 o'clock!

The next step would seem to be - perhaps at the beginning of secondary school - an attempt to integrate regions with rotations, making it clear that the standard angle diagram can represent both concepts. A possible vehicle for this step is the discussion of rotational symmetry, which can be equivalently treated using repeated regions or repeated rotations. Only after this attempt at integration should angle measurement be treated, at best starting with non-standard units and ending with the circular protractor.

One final implication: the formulation of a single, mathematically rigorous definition of angle should follow much later. In view of the variety of practical applications of angles, it is doubtful whether any exclusive definition could serve a useful purpose outside a strict axiomatic treatment.

RESEARCH NEEDS

My review of research on the angle concept has revealed astonishing gaps in our knowledge. For example, why do children have so much difficulty comparing the angles in Fig 1? Some interesting questions are the following.

- When and under what circumstances do children conserve angle? How are conservation judgements affected by the transformation made and by the presence or absence of a demonstration of this transformation? Is there a hierarchy for the discrimination and recognition of angles, depending on the transformation involved and the size of the angle?

- How does knowledge of rotation and its representation develop?

- What factors affect ability to compare angles by superposition?

- How do children spontaneously classify angles? What roles do size, orientation, arm length, etc. play?

- How does the wording of the question ("Are these the same?", "Are these the same angle?", "Which of these is sharper?", "Which bends the most?") affect performance on angle comparison tasks?

- When and how do children begin to integrate the various angle concepts?

Finally there is a host of questions on teaching angles: How to assist the abstraction of angles as regions, how to promote integration, when to teach measurement, what definition (if any one) is most effective, and so on.

This is an immense research program. I hope that this survey will lead more mathematics educators to enter this field with enthusiasm and creativity.

REFERENCES

Beilin, H. (1979) *Stategies and structures in understanding geometry.* New York: City University of New York. ERIC document no. SED 79 12809.

Bryant, P. (1974) *Perception and understanding in young children.* London: Methuen.

Close, G.S. (1982) *Children's understanding of angle at the primary/ secondary transfer stage.* London: Polytechnic of the South Bank.

Kirsche, P. (1987) Zur Behandlung von Drehsymmetrie und Drehung in Geometrieunterricht der Grundschule. *Sachunterricht und Mathematik in der Primarstufe* 15, 222-227.

Noss, R. (1987) Children's learning of geometrical concepts through LOGO. *Journal for Research in Mathematics Education* 18, 343-362.

Piaget, J., Inhelder, B., & Szeminska, A. (1960) *The child's conception of geometry.* London: Routledge & Kegan Paul.

Wallrabenstein, H. (1973) Development and signification of a geometry test. *Educational Studies in Mathematics* 5, 81-89.